JEE ADVANCED

Single Correct Answer Type

1. The value of the definite integral $\int_0^1 \left(1+e^{-x^2}\right) dx$ is
   a. -1 b. 2
   c. $1+e^{-1}$ d. none of these
   (IIT-JEE 1981)

2. Let a, b, c be nonzero real numbers such that
   $\int_0^1 \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx$
   $= \int_0^2 \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx.$
   Then, the quadratic equation $ax^2+bx+c=0$ has
   a. no root in (0, 2)
   b. at least one root in (0, 2)
   c. a double root in (0, 2)
   d. two imaginary roots
   (IIT-JEE 1981)

3. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx$ is
   a. $\pi/4$ b. $\pi/2$
   c. π d. none of these
   (IIT-JEE 1983)

4. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\,dx$ has the value
   a. π b. 1
   c. 0 d. none of these
   (IIT-JEE 1985)

5. Let $f: R \to R$ and $g: R \to R$ be continuous functions. Then the value of the integral
   $\int_{-\pi/2}^{\pi/2} [f(x)+f(-x)]\,[g(x)-g(-x)]\,dx$ is
   a. π b. 1
   c. -1 d. 0 **(IIT-JEE 1990)**

6. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is
   a. 0 b. 1
   c. $\pi/2$ d. $\pi/4$ **(IIT-JEE 1993)**

7. If $f(x) = A\sin\left(\frac{\pi x}{2}\right)+B$, $f'\left(\frac{1}{2}\right)=\sqrt{2}$, and $\int_0^1 f(x)\,dx = \frac{2A}{\pi}$, then constants A and B are
   a. $\frac{\pi}{2}$ and $\frac{\pi}{2}$ b. $\frac{2}{\pi}$ and $\frac{3}{\pi}$
   c. 0 and $\frac{-4}{\pi}$ d. $\frac{4}{\pi}$ and 0
   (IIT-JEE 1995)

8. The value of $\int_0^{2\pi} [2\sin x]\,dx$, where [.] represents the greatest integral function, is
   a. $\frac{-5\pi}{3}$ b. $-\pi$ c. $\frac{5\pi}{3}$ d. -2π
   (IIT-JEE 1995)

9. Let f be a positive function. Let $I_1 = \int_{1-k}^{k} x f[x(1-x)]\, dx$, $I_2 = \int_{1-k}^{k} f[x(1-x)]\, dx$, where $2k-1>0$. Then $\frac{I_1}{I_2}$ is

a. 2 b. k c. $\frac{1}{2}$ d. 1

(IIT-JEE 1997)

10. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x+\pi)$ equals

a. $g(x) + g(\pi)$ b. $g(x) - g(\pi)$

c. $g(x)\, g(\pi)$ d. $\frac{g(x)}{g(\pi)}$ **(IIT-JEE 1997)**

11. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to

a. 2 b. -2 c. 1/2 d. $-1/2$

(IIT-JEE 1999)

12. If for a real number y, $[y]$ is the greatest integral function less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx$ is

a. $-\pi$ b. 0 c. $-\pi/2$ d. $\pi/2$

(IIT-JEE 1999)

13. Let $g(x) = \int_0^x f(t)\, dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$, for $t \in [0, 1]$, and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$. Then $g(2)$ satisfies the inequality

a. $-\frac{3}{2} \le g(2) < \frac{1}{2}$ b. $\frac{1}{2} \le g(2) \le \frac{3}{2}$

c. $\frac{3}{2} < g(2) \le \frac{5}{2}$ d. $2 < g(2) < 4$

(IIT-JEE 2000)

14. If $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)\, dx =$

a. 0 b. 1 c. 2 d. 3

(IIT-JEE 2000)

15. The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is

a. 3/2 b. 5/2 c. 3 d. 5

(IIT-JEE 2000)

16. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\, dx$, where $a > 0$, is

a. π b. $a\pi$

c. $\pi/2$ d. 2π **(IIT-JEE 2001)**

17. Let $f(x) = \int_1^x \sqrt{2-t^2}\, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

a. ± 1 b. $\pm \frac{1}{\sqrt{2}}$

c. $\pm \frac{1}{2}$ d. 0 and 1

(IIT-JEE 2002)

18. Let $T > 0$ be a fixed real number. Suppose f is continuous function such that for all $x \in R$, $f(x+T) = f(x)$. If $I = \int_0^T f(x)\, dx$, then the value of $\int_3^{3+3T} f(2x)\, dx$ is

a. $3/2I$ b. $2I$

c. $3I$ d. $6I$ **(IIT-JEE 2002)**

19. The integral $\int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1+x}{1-x}\right)\right) dx$ is equal to (where $[\cdot]$ represents the greatest integer function)

a. $-\frac{1}{2}$ b. 0 c. 1 d. $2\ln\left(\frac{1}{2}\right)$

(IIT-JEE 2002)

20. If $L(m, n) = \int_0^1 t^m (1+t)^n\, dt$, then the expression for $L(m, n)$ in terms of $L(m+1, n-1)$ is, $(m, n \in N)$,

a. $\frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)$

b. $\frac{n}{m+1} L(m+1, n-1)$

c. $\frac{2^n}{m+1} + \frac{n}{m+1} L(m+1, n-1)$

d. $\frac{m}{n+1} L(m+1, n-1)$ **(IIT-JEE 2003)**

21. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\, dt$, then $f(x)$ increases in

a. (0, 2) b. no value of x

c. $(0, \infty)$ d. $(-\infty, 0)$ **(IIT-JEE 2003)**

22. If $f(x)$ is differentiable and $\int_0^{t^2} x f(x)\, dx = \frac{2}{5} t^5$, then $f\left(\frac{4}{25}\right)$ equals

a. 2/5 b. $-5/2$

c. 1 d. 5/2 **(IIT-JEE 2004)**

23. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ is

a. $\frac{\pi}{2} + 1$ b. $\frac{\pi}{2} - 1$

c. -1 d. 1 **(IIT-JEE 2004)**

24. $\int_{-2}^{0} \{x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)\}\, dx$ is equal to

a. -4 b. 0

c. 4 d. 6 **(IIT-JEE 2005)**

25. $\lim_{x \to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}}$ equals

a. $\frac{8}{\pi} f(2)$ b. $\frac{2}{\pi} f(2)$

c. $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ d. $4f(2)$ **(IIT-JEE 2007)**

26. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-(f'(t))^2}\, dt = \int_0^x f(t)dt, 0 \le x \le 1$, and $f(0) = 0$, then

a. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

b. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

c. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

d. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ **(IIT-JEE 2009)**

27. The value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are)

a. $\frac{22}{7} - \pi$ b. $\frac{2}{105}$ c. 0 d. $\frac{71}{15} - \frac{3\pi}{2}$ **(IIT-JEE 2010)**

28. The value of $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)}{t^4+4} dt$ is

a. 0 b. $\frac{1}{12}$

c. $\frac{1}{24}$ d. $\frac{1}{64}$ **(IIT-JEE 2010)**

29. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

a. 1 b. 1/3

c. 1/2 d. $1/e$ **(IIT-JEE 2010)**

30. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is

a. $\frac{1}{4} \ln \frac{3}{2}$ b. $\frac{1}{2} \ln \frac{3}{2}$

c. $\ln \frac{3}{2}$ d. $\frac{1}{6} \ln \frac{3}{2}$

(IIT-JEE 2011)

31. Let $f: [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then

a. $R_1 = 2R_2$ b. $R_1 = 3R_2$

c. $2R_1 = R_2$ d. $3R_1 = R_2$

(IIT-JEE 2011)

32. The value of the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log_e \frac{\pi+x}{\pi-x}\right) \cos x\, dx$ is

a. 0 b. $\frac{\pi^2}{2} - 4$ c. $\frac{\pi^2}{2} + 4$ d. $\frac{\pi^2}{2}$

(IIT-JEE 2012)

33. Let $f: \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant, and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

a. $(2e-1, 2e)$ b. $(e-1, 2e-1)$

c. $\left(\frac{e-1}{2}, e-1\right)$ d. $\left(0, \frac{e-1}{2}\right)$

(JEE Advanced 2013)

34. Let $f: [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\, dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

a. $e^2 - 1$ b. $e^4 - 1$

c. $e - 1$ d. e^4

(JEE Advanced 2014)

35. The following integral $\int_{\pi/4}^{\pi/2} (2 \operatorname{cosec} x)^{17} dx$ is equal to

a. $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$

b. $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{17} du$

c. $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{17} du$

d. $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$ **(JEE Advanced 2014)**

36. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are

a. $m = 13, M = 24$ b. $m = \frac{1}{4}, M = \frac{1}{2}$

c. $m = -11, M = 0$ d. $m = 1, M = 12$

(JEE Advanced 2015)

Multiple Correct Answer Type

1. If $\int_0^x f(t)dt = x + \int_x^1 t f(t)dt$, then the value of $f(1)$ is

a. 1/2 b. 0 c. 1 d. −1/2

(IIT-JEE 1998)

2. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then $\int_{-1}^{1} f(x)\,dx$ is

a. 1 b. 2 c. 0 d. 1/2

(IIT-JEE 1998)

3. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for $n = 1, 2, 3, \ldots$. Then

a. $S_n < \frac{\pi}{3\sqrt{3}}$ b. $S_n > \frac{\pi}{3\sqrt{3}}$

c. $T_n < \frac{\pi}{3\sqrt{3}}$ d. $T_n > \frac{\pi}{3\sqrt{3}}$

(IIT-JEE 2008)

4. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then

a. $f'(x)$ vanishes at least twice on $[0, 1]$

b. $f'\left(\frac{1}{2}\right) = 0$

c. $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x\,dx = 0$

d. $\int_0^{1/2} f(t)e^{\sin \pi t}\,dt = \int_{1/2}^{1} f(1-t)\,e^{\sin \pi t}\,dt$ **(IIT-JEE 2008)**

5. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$, $n = 0, 1, 2, \ldots$, then

a. $I_n = I_{n+2}$ b. $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

c. $\sum_{m=1}^{10} I_{2m} = 0$ d. $I_n = I_{n+1}$

(IIT-JEE 2009)

6. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true?

a. $f''(x)$ exists for all $x \in (0, \infty)$.

b. $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$.

c. There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

d. There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$.

(IIT-JEE 2010)

7. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$. Then

a. $S \ge \frac{1}{e}$ b. $S \ge 1 - \frac{1}{e}$

c. $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ d. $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

(IIT-JEE 2012)

8. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then

a. f has a local maximum at $x = 2$

b. f is decreasing on $(2, 3)$

c. there exists some $c \in (0, \infty)$ such that $f''(c) = 0$

d. f has a local minimum at $x = 3$ **(IIT-JEE 2012)**

9. For $a \in R$ (the set of all real numbers), $a \ne -1$,

$$\lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]} = \frac{1}{60}.$$

Then $a =$

a. 5 b. 7 c. $\frac{-15}{2}$ d. $\frac{-17}{2}$

(JEE Advanced 2013)

10. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)dt & \text{if } a \le x \le b, \\ \int_a^b f(t)dt & \text{if } x > b \end{cases}$$

Then

a. $g(x)$ is continuous but not differentiable at a

b. $g(x)$ is differentiable on R

c. $g(x)$ is continuous but not differentiable at b

d. $g(x)$ is continuous and differentiable at either a or b but not both

(JEE Advanced 2014)

11. Let $f : (0, \infty) \to R$ be given by $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$, then

a. $f(x)$ is monotonically increasing on $[1, \infty)$

b. $f(x)$ is monotonically decreasing on $(0, 1)$

c. $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

d. $f(2^x)$ is an odd function of x on R

(JEE Advanced 2014)

12. The option(s) with the values of a and L that satisfy the following equation is (are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L?$$

a. $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ b. $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

c. $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ d. $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(JEE Advanced 2015)

13. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

a. $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ b. $\int_0^{\pi/4} f(x)dx = 0$

c. $\int_0^{\pi/4} xf(x) = \frac{1}{6}$ d. $\int_0^{\pi/4} f(x)dx = 1$

(JEE Advanced 2015)

Linked Comprehension Type

For Problem 1-3

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$. For more accurate result, for $c \in (a, b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$.

(IIT-JEE 2008)

1. $\int_0^{\pi/2} \sin x dx$ is equal to

a. $\frac{\pi}{8}(1+\sqrt{2})$ b. $\frac{\pi}{4}(1+\sqrt{2})$

c. $\frac{\pi}{8\sqrt{2}}$ d. $\frac{\pi}{4\sqrt{2}}$

2. If $\lim\limits_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

a. 4 b. 3 c. 2 d. 1

3. If $f''(x) < 0\ \forall\ x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

a. $\frac{f(b)-f(a)}{b-a}$ b. $\frac{2(f(b)-f(a))}{b-a}$

c. $\frac{2f(b)-f(a)}{2b-a}$ d. 0

For Problem 4-5

Consider the function $f: (-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \frac{x^2-ax+1}{x^2+ax+1}, 0 < a < 2$.

4. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}dt$ which of the following is true?

a. $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
b. $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
c. $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
d. $g'(x)$ does not change sign on $(-\infty, \infty)$

(IIT-JEE 2008)

5. Let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log_e t\right)f(t)\,dt$ for all $x \in (1, \infty)$.

Then which of the following is true?

a. g is increasing on $(1, \infty)$
b. g is decreasing on $(1, \infty)$
c. g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
d. g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

(IIT-JEE 2012)

For Problems 6 and 7

Given that for each $a \in (0, 1)$, $\lim\limits_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. **(JEE Advanced 2014)**

6. The value of $g\left(\frac{1}{2}\right)$ is

a. π b. 2π c. $\frac{\pi}{2}$ d. $\frac{\pi}{4}$

7. The value of $g'\left(\frac{1}{2}\right)$ is

a. $\frac{\pi}{2}$ b. π c. $-\frac{\pi}{2}$ d. 0

For Problem 8 and 9

Let $F: R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$. **(JEE Advanced 2015)**

8. The correct statement(s) is (are)

a. $f'(1) < 0$
b. $f(2) < 0$
c. $f'(x) \neq 0$ for any $x \in (1, 3)$
d. $f'(x) = 0$ for some $x \in (1, 3)$

9. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, then the correct expression(s) is (are)

a. $9f'(3) + f'(1) - 32 = 0$ b. $\int_1^3 f(x)dx = 12$

c. $9f'(3) - f'(1) + 32 = 0$ d. $\int_1^3 f(x)dx = -12$

Matching Column Type

1. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(i) Two rays in the first quadrant $x + y = \|a\|$ and $ax - y = 1$ intersects each other in the interval $a \in (a_0, \infty)$, the value of a_0 is	(a) 2
(ii) Point (α, β, γ) lies on the plane $x + y + z = 2$. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\hat{k} \times (\hat{k} \times \vec{a}) = 0$, then $\gamma =$	(b) 4/3
(iii) $\left\|\int_0^1 (1-y^2)dy\right\| + \left\|\int_1^0 (y^2-1)dy\right\|$	(c) $\left\|\int_0^1 \sqrt{1-x}dx\right\| + \left\|\int_{-1}^0 \sqrt{1+x}dx\right\|$
(iv) If $\sin A \sin B \sin C + \cos A \cos B = 1$, then the value of $\sin C =$	(d) 1

(IITJEE 2006)

2. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(i) $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \log(\sin x)^{\sin x})dx$	(a) 1
(ii) Area bounded by $-4y^2 = x$ and $x - 1 = 5y^2$	(b) 0
(iii) Cosine of the angle of intersection of curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is	(c) 6 ln 2
	(d) 4/3

(IITJEE 2006)

3. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(a) $\int_{-1}^{1} \frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\frac{2}{3}\right)$
(c) $\int_2^3 \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(d) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

(IIT-JEE 2007)

4. Match the statements/expressions in Column I with the open intervals in Column II.

Column I	Column II
(a) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(b) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(c) Interval in which at least one of the points locus maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(d) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

(IIT-JEE 2009)

5. Match the statements given in Column I with the values given in Column II.

Column I	Column II
(a) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(b) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(c) The value of $\frac{\pi^2}{\log_e 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is	(r) $\frac{\pi}{3}$
(d) The maximum value of $\left\|\text{Arg}\left(\frac{1}{1-z}\right)\right\|$ for $\|z\| = 1, z \neq 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

(IIITJEE 2011)

6. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(p) The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	(1) 8

(q) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(2) 2
(r) $\int_{-2}^{2} \frac{3x^2}{1+e^x} dx$ equals	(3) 4
(s) $\frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}{\int_{0}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}$ equals	(4) 0

(JEE Advanced 2014)

Codes:

	(p)	(q)	(r)	(s)
a.	(3)	(2)	(4)	(1)
b.	(2)	(3)	(4)	(1)
c.	(3)	(2)	(1)	(4)
d.	(2)	(3)	(1)	(4)

Integer Answer Type

1. Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)\,dt$. Then the value of $f(\ln 5)$ is **(IIT-JEE 2009)**

2. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real-valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$ is

(IIT-JEE 2010)

3. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2}(1-x^2)^5 \right\} dx$ is

(JEE Advanced 2014)

4. Let $f: R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\,|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **(JEE Advanced 2015)**

5. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right) dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is

(JEE Advanced 2015)

6. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

(JEE Advanced 2015)

7. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$ where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)} dx$, then the value of $(4I - 1)$ is **(JEE Advanced 2015)**

Fill in the Blanks Type

1. $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x) dx =$ ______. **(IIT-JEE 1987)**

2. The integral $\int_0^{1.5} [x^2]\, dx$, where $[\cdot]$ denotes the greatest integer function, equals ______. **(IIT-JEE 1988)**

3. The value of $\int_{-2}^{2} |1-x^2|\, dx$ is ______. **(IIT-JEE 1989)**

4. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi$ is ______. **(IIT-JEE 1993)**

5. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$ is ______. **(IIT-JEE 1994)**

6. If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \ne b$, then $\int_1^2 f(x)\, dx =$ ______. **(IIT-JEE 1996)**

7. For $n \in N$, $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx =$ ______. **(IIT-JEE 1996)**

8. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is ______. **(IIT-JEE 1997)**

9. Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}\,dx=F(k)-F(1)$, then one of the possible values of k is ______ .

(IIT-JEE 1997)

True/False Type

1. The value of the integral $\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ is equal to a.

(IIT-JEE 1988)

Subjective Type

1. Show that $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)=\log 6$.

(IIT-JEE 1981)

2. Evaluate $\int_0^1 (tx+1-x)^n\,dx$, where n is a positive integer and t is a parameter independent of x. Hence, show that $\int_0^1 x^k(1-x)^{n-k}\,dx=[{}^nC_k(n+1)]^{-1}$ for $k=0, 1, \ldots, n$.

(IIT-JEE 1981)

3. Show that $\int_0^\pi xf(\sin x)dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$.

(IIT-JEE 1982)

4. Find the value of $\int_{-1}^{3/2}|x\sin\pi x|dx$.

(IIT-JEE 1982)

5. Evaluate $\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$. **(IIT-JEE 1983)**

6. Evaluate $\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$. **(IIT-JEE 1984)**

7. Given a function $f(x)$ such that
   a. it is integrable over every interval on the real line, and
   b. $f(t+x)=f(x)$, for every x and a real t.

   Then show that the integral $\int_a^{a+t}f(x)dx$ is independent of a.

8. Evaluate $\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$. **(IIT-JEE 1985)**

9. Evaluate $\int_0^\pi \frac{x\,dx}{1+\cos\alpha\sin x}$, where $0<\alpha<\pi$.

(IIT-JEE 1986)

10. If f and g are continuous functions on $[0, a]$ satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=2$, then show that $\int_0^a f(x)g(x)dx=\int_0^a f(x)\,dx$. **(IIT-JEE 1989)**

11. Show that $\int_0^{\pi/2}f(\sin 2x)\sin x\,dx$ $=\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx$. **(IIT-JEE 1990)**

12. Prove that for any positive integer k,

$$\frac{\sin 2kx}{\sin x}=2[\cos x+\cos 3x+\cdots+\cos(2k-1)x].$$

Hence, prove that $\int_0^{\pi/2}\sin 2xk\cot x\,dx=\frac{\pi}{2}$.

(IIT-JEE 1990)

13. If f is a continous function with $\int_0^x f(t)dt\to\infty$ as $|x|\to\infty$, then show that every line $y=mx$ intersects the curve $y^2+\int_0^x f(t)dt=2$. **(IIT-JEE 1991)**

14. Evaluate $\int_0^\pi \frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$. **(IIT-JEE 1991)**

15. Determine a positive integer $n\le 5$ such that $\int_0^1 e^x(x-1)^n=16-6e$. **(IIT-JEE 1992)**

16. Evaluate $\int_2^3\frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$.

(IIT-JEE 1993)

17. Show that $\int_0^{n\pi+v}|\sin x|dx=2n+1-\cos v$, where n is a positive integer and $0\le v<\pi$. **(IIT-JEE 1994)**

18. If $U_n=\int_0^\pi\frac{1-\cos nx}{1-\cos x}dx$, where n is positive integer or zero, then show that $U_{n+2}+U_n=2U_{n+1}$. Hence, deduce that $\int_0^{\pi/2}\frac{\sin^2 n\theta}{\sin^2\theta}=\frac{1}{2}n\pi$. **(IIT-JEE 1995)**

19. Evaluate the definite integral

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx.$$

(IIT-JEE 1995, 1996)

20. Evaluate $\int_0^{\pi/4}\ln(1+\tan x)dx$. **(IIT-JEE 1995, 1996)**

21. Let $a+b=4$, where $a<2$, and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)\,dx+\int_0^b g(x)dx$ increases as $(b-a)$ increases.

(IIT-JEE 1997)

22. Determine the value of $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$.

(IIT-JEE 1997)

23. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x dx$.

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2)dx$. **(IIT-JEE 1998)**

24. For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t} dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and find the value of $f(e) + f\left(\frac{1}{e}\right)$.

(IIT-JEE 2000)

25. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$, then find $\frac{dy}{dx}$ at $x = \pi$.

(IIT-JEE 2004)

26. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x| + \frac{\pi}{3}\right)} dx$.

(IIT-JEE 2004)

27. Evaluate $\int_0^\pi e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\, dx$.

(IIT-JEE 2005)

28. Evaluate $5050\frac{\int_0^1 (1-x^{50})^{100} dx}{\int_0^1 (1-x^{50})^{101} dx}$. **(IIT-JEE 2006)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. d.	**2.** b.	**3.** a.	**4.** c.
5. d.	**6.** d.	**7.** d.	**8.** b.
9. c.	**10.** a.	**11.** a.	**12.** c.
13. b.	**14.** c.	**15.** b.	**16.** c.
17. a.	**18.** c.	**19.** a.	**20.** a.
21. d.	**22.** a.	**23.** b.	**24.** c.
25. a.	**26.** c.	**27.** a.	**28.** b.
29. b.	**30.** a.	**31.** c.	**32.** b.
33. d.	**34.** b.	**35.** a.	**36.** d.

Multiple Correct Answers Type

1. a.	**2.** a.	**3.** a., d.	**4.** a., b., c., d.
5. a., b., c.	**6.** b., c.	**7.** a., b., d.	**8.** a., b., c., d.
9. b., d.	**10.** a., c.	**11.** a., c., d.	**12.** a., c.
13. a., b.			

Linked Comprehension Type

1. a.	**2.** d.	**3.** b.	**4.** b.
5. b.	**6.** a.	**7.** d.	
8. a., b., c.	**9.** c., d.		

Matching Column Type

1. (iii) – (b), (c)
2. (i) – (a)
3. (a) – (s); (b) – (s); (c) – (p); (d) – (r).
4. (b) – (p), (t)
5. (b) – (p); (c) – (s)
6. d.

Integer Answer Type

1. 0	**2.** 4	**3.** 2	**4.** 7
5. 9	**6.** 3	**7.** 0	

Fill in the Blanks Type

1. $-\left(\frac{15\pi - 32}{60}\right)$	**2.** $2-\sqrt{2}$
3. 4	**4.** $\pi(\sqrt{2}-1)$
5. $\frac{1}{2}$	**6.** $\frac{1}{a^2-b^2}\left[a\log 2 - 5a + \frac{7b}{2}\right]$
7. π^2	**8.** 2
9. 16	

True/False Type

1. True

Subjective Type

2. $\frac{t^{n+1}-1}{(t-1)(n+1)}$	**4.** $\frac{3}{\pi} + \frac{1}{\pi^2}$
5. $\frac{1}{20}\log 3$	**6.** $\frac{6-\pi\sqrt{3}}{12}$
8. $\frac{\pi^2}{16}$	**9.** $\frac{\pi\alpha}{\sin\alpha}$

14. $\frac{8}{\pi^2}$

15. $n = 3$

16. $\frac{3}{2}\log 2 - \frac{1}{10}$

17. $2n + 1 - \cos v$

19. $-\frac{\pi}{\sqrt{3}} - \frac{\pi}{4}\log_e\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| + \frac{\pi^2}{12}$

20. $\frac{\pi}{8}\ln 2$

22. π^2

23. $\log 2$

24. $\frac{1}{2}(\log x)^2$

25. 2π

26. $\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3 - \frac{\pi}{4}\right]$

27. $\frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{1}{2}e\sin\left(\frac{1}{2}\right) - 1\right]$

28. 5051

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. d. $\int_0^1\left(1+e^{-x^2}\right)dx$

$$=\int_0^1\left(1+1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots\infty\right)dx$$

$$=\left[2x-\frac{x^3}{3.1!}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\cdots\infty\right]_0^1$$

$$=\left[2-\frac{1}{3.1!}+\frac{1}{5.2!}-\frac{1}{7.3!}+\cdots\infty\right]$$

Clearly, d is the correct alternative.

2. b. Let $f(x)=\int(1+\cos^8 x)(ax^2+bx+c)dx$

$\therefore f'(x)=(1+\cos^8 x)(ax^2+bx+c)$ (1)

From the given conditions,

$f(1)-f(0)=0$ or $f(0)=f(1)$ (2)

and $f(2)-f(0)=0$ or $f(0)=f(2)$ (3)

From equations (2) and (3), we get $f(0)=f(1)=f(2)$.

By Rolle's theorem for $f(x)$ in [0, 1]: $f'(\alpha)=0$, $\exists$ at least one α such that $0<\alpha<1$.

By Rolle's theorem for $f(x)$ in [1, 2]: $f'(\beta)=0$, $\exists$ at least one β such that $1<\beta<2$.

Now, from equation (1), $f'(\alpha)=0$

or $(1+\cos^8\alpha)(a\alpha^2+b\alpha+c)=0$ $\quad(\because 1+\cos^8\alpha\neq 0)$

or $a\alpha^2+b\alpha+c=0$

i.e., α is a root of the equation $ax^2+bx+c=0$.

Similarly, β is a root of the equation $ax^2+bx+c=0$.

But equation $ax^2+bx+c=0$ being a quadratic equation cannot have more than two roots.

Hence, equation $ax^2+bx+c=0$ has one root α between 0 and 1, and other root β between 1 and 2.

3. a. $I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ (1)

$\therefore I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$ (2)

[Using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$]

Adding equations (1) and (2), we get $2I=\int_0^{\pi/2}1dx$

$\therefore I=\pi/4$

4. c. $I=\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx,\ n\in Z$ (1)

$$=\int_0^{\pi}e^{\cos^2(\pi-x)}\cos^3[(2n+1)(\pi-x)]\,dx$$

[Using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$]

$$=\int_0^{\pi}e^{\cos^2 x}\cos^3[(2n+1)\pi-(2n+1)x]\,dx$$

$$=-\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx$$

$$=-I$$

$\therefore I=0$

5. d. Since $h(x)=(f(x)+f(-x))(g(x)-g(-x))$

$\therefore h(-x)=(f(-x)+f(x))(g(-x)-g(x))$

$\therefore h(-x)=-h(x)$

$h(x)$ is odd function.

$\therefore \int_{-\pi/2}^{\pi/2}(f(x)+f(-x))(g(x)-g(-x))dx=0$

6. d. Let $I=\int_0^{\pi/2}\frac{dx}{1+\tan^3 x}$

$$=\int_0^{\pi/2}\frac{\cos^3 x}{\sin^3 x+\cos^3 x}dx \quad (1)$$

$$=\int_0^{\pi/2}\frac{\cos^3\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\pi/2}\frac{\sin^3 x}{\cos^3 x+\sin^3 x}dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I=\int_0^{\pi/2}1dx$$

$\therefore I=\frac{\pi}{4}$

7. d. $f(x)=A\sin(\pi x/2)+B$

$\therefore f'(x)=\frac{A\pi}{2}\cos\left(\frac{\pi x}{2}\right)$

$\therefore f'\left(\frac{1}{2}\right)=\frac{A\pi}{2}\cos\frac{\pi}{4}=\sqrt{2}$ (given)

$\therefore A=4/\pi$

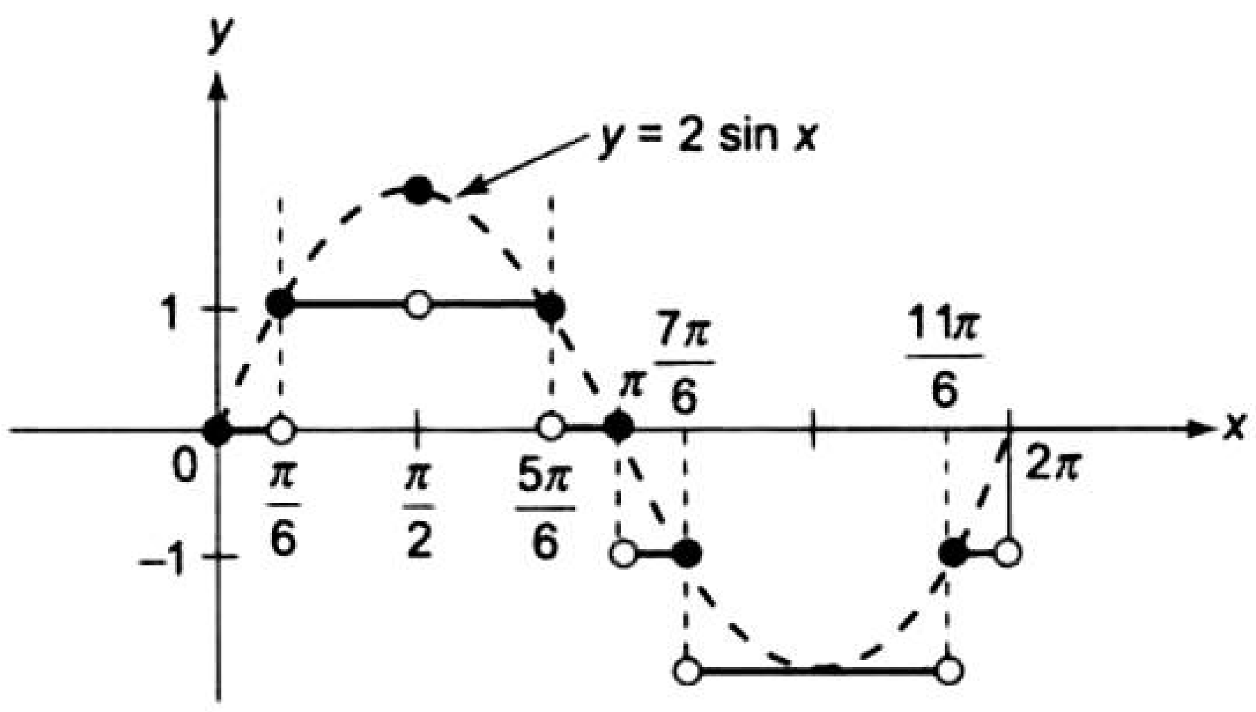
Also, given $\int_0^1 f(x)\,dx=\frac{2A}{\pi}$

$$\therefore \int_0^1\left[A\sin\left(\frac{\pi x}{2}\right)+B\right]dx=\frac{2A}{\pi}$$

$$\therefore \left|-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right)+Bx\right|_0^1=\frac{2A}{\pi}$$

$\therefore B+\frac{2A}{\pi}=\frac{2A}{\pi}\quad \therefore B=0$

8. b. $I=\int_0^{2\pi}[2\sin x]dx$



From the graph,

$$I=\int_{\pi/6}^{5\pi/6} 1dx+\int_{\pi}^{7\pi/6} -1\,dx+\int_{7\pi/6}^{11\pi/6} -2\,dx+\int_{11\pi/6}^{2\pi} -1dx$$

$$=\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)+\left(-\frac{7\pi}{6}+\pi\right)+2\left(-\frac{11\pi}{6}+\frac{7\pi}{6}\right)+\left(-2\pi+\frac{11\pi}{6}\right)$$

$$=\frac{2\pi}{3}-\frac{\pi}{6}-\frac{8\pi}{6}-\frac{\pi}{6}=-\pi$$

9. c. Given f is a positive function, and

$$I_1=\int_{1-k}^{k} x f[x(1-x)]\,dx$$

$$I_2=\int_{1-k}^{k} f[x(1-x)]dx$$

Now, $I_1=\int_{1-k}^{k} xf[x(1-x)]dx$ (1)

$=\int_{1-k}^{k}(1-x)f[(1-x)x]dx$ (2)

$\left[\text{Using the property } \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

Adding equations (1) and (2), we get

$$2I_1=\int_{1-k}^{k} f[x(1-x)]dx=I_2 \text{ or } \frac{I_1}{I_2}=\frac{1}{2}$$

10. a. $g(x)=\int_0^x \cos^4 t\,dt$

$\therefore g(x+\pi)=\int_0^{x+\pi}\cos^4 t\,dt$

$=\int_0^x \cos^4 t\,dt+\int_x^{x+\pi}\cos^4 t\,dt$

$=g(x)+\int_0^{\pi}\cos^4 t\,dt$ [$\because$ period of $\cos^4 t$ is π]

$=g(x)+g(\pi)$

11. a. $I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ (1)

$=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos(\pi-x)}$

$\left[\text{Using the property } \int_a^b f(x)\,dx=\int_a^b (f(a+b-x))dx\right]$

$=\int_{\pi/4}^{3\pi/4}\frac{dx}{1-\cos x}$ (2)

Adding (1) and (2), we get

$$2I=\int_{\pi/4}^{3\pi/4}\left(\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\right)dx$$

$$=\int_{\pi/4}^{3\pi/4} 2\,\text{cosec}^2 x\,dx$$

$$=2(-\cot x)_{\pi/4}^{3\pi/4}$$

$$=-2[\cot 3\pi/4-\cot \pi/4]$$

$$=-2(-1-1)=4$$

$\therefore I=2$

12. c. Refer to graph of **Question 8**. Then

$$\int_{\pi/2}^{3\pi/2}[2\sin x]dx$$

$$=\int_{\pi/2}^{5\pi/6} 1dx+\int_{\pi}^{7\pi/6} -1dx+\int_{7\pi/6}^{3\pi/2} -2dx$$

$$=\left[\frac{5\pi}{6}-\frac{\pi}{2}\right]-\left[\frac{7\pi}{6}-\pi\right]-2\left[\frac{3\pi}{2}-\frac{7\pi}{6}\right]$$

$$=\frac{-\pi}{2}$$

13. b. $g(x)=\int_0^x f(t)dt,$

$$\therefore\ g(2)=\int_0^2 f(t)=\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt$$

Now, $\frac{1}{2}\le f(t)\le 1$ for $t\in[0,1]$

$$\therefore\ \int_0^1\frac{1}{2}dt\le\int_0^1 f(t)\,dt\le\int_0^1 1\,dt$$

or $\frac{1}{2}\le\int_0^1 f(t)\,dt\le 1$ (1)

Again, $0\le f(t)\le\frac{1}{2}$ for $t\in[1,2]$

$$\therefore\ \int_1^2 0\,dt\le\int_1^2 f(t)\le\int_1^2\frac{1}{2}dt$$

$\therefore 0\le\int_1^2 f(t)\,dt\le\frac{1}{2}$ (2)

From equations (1) and (2), we get

$$\frac{1}{2}\le\int_0^1 f(t)+\int_1^2 f(t)\,dt\le\frac{3}{2}$$

$$\therefore\ \frac{1}{2}\le g(2)\le\frac{3}{2}$$

14. c. If $f(x)=\begin{cases} e^{\cos x}\sin x, & \text{for } |x|\le 2\\ 2, & \text{otherwise}\end{cases}$

or $\int_{-2}^{3} f(x)\,dx=\int_{-2}^{2} f(x)\,dx+\int_2^3 f(x)\,dx$

$$=\int_{-2}^{2} e^{\cos x}\sin x\,dx+\int_2^3 2\,dx=0+2[x]_2^3=2$$

[$\because e^{\cos x}\sin x$ is an odd function]

15. b. Let $I=\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$

For $\frac{1}{e}<x<1$, $\log_e x<0$. Hence, $\frac{\log_e x}{x}<0$.

For $1 < x < e^2$, $\log x > 0$. Hence, $\frac{\log_e x}{x} > 0$.

$$\therefore\ I = \int_{1/e}^{1} -\frac{\log_e x}{x}\,dx + \int_{1}^{e^2} \frac{\log_e x}{x}\,dx$$

$$= -\frac{1}{2}\left[(\log_e x)^2\right]_{1/e}^{1} + \frac{1}{2}\left[(\log_e x)^2\right]_{1}^{e^2}$$

$$= -\frac{1}{2}\left[0-(-1)^2\right] + \frac{1}{2}\left[(2)^2 - 0\right]$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

16. c. $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$ (1)

$$= \int_{-\pi}^{\pi} \frac{\cos^2 (0-x)}{1+a^{(0-x)}}\,dx$$

[Using the property $\int_a^b f(x)\,dx = \int_a^b (f(a+b-x))\,dx$]

$\therefore\ I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x}\,dx$ (2)

Adding equations (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x\,dx$$

$$= 2\int_0^{\pi} \cos^2 x\,dx$$

$\therefore\ 2I = 4\int_0^{\pi/2} \cos^2 x\,dx$ (3)

$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$

$= 4\int_0^{\pi/2} \sin^2 x\,dx$ (4)

Adding equations (3) and (4), we get

$$4I = 4\int_0^{\pi/2} 1\,dx$$

or $I = \pi/2$

17. a. Here, $f(x) = \int_1^x \sqrt{2-t^2}\,dt$

$\therefore f'(x) = \sqrt{2-x^2}$

Now, the given equation $x^2 - f'(x) = 0$ becomes

$x^2 - \sqrt{2-x^2} = 0$

or $x^2 = \sqrt{2-x^2}$

$\therefore\ x = \pm 1$

18. c. Let $I_1 = \int_3^{3+3T} f(2x)\,dx$.

Put $2x = y$, so that $I_1 = \frac{1}{2}\int_6^{6+6T} f(y)\,dy$

$= \frac{1}{2} 6\int_0^T f(y)\,dy$ $[\because f(x)$ has period $T]$

$= 3I$

19. a. $I = \int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx$

$$= \int_{-1/2}^{1/2} [x]\,dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right)dx$$

$$= \int_{-1/2}^{0} -1\,dx + \int_0^{1/2} 0\,dx + 0$$

$\left[\because \log\left(\frac{1+x}{1-x}\right) \text{ is an odd function}\right]$

$$= [-x]_{-1/2}^{0} = 0 - \left(\frac{1}{2}\right) = -1/2$$

20. a. Given $L(m, n) = \int_0^1 t^m (1+t)^n\,dt$

Integrating by parts considering $(1+t)^n$ as first function, we get

$$L(m, n) = \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}\,dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)$$

21. d. We have $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$

$$\therefore f'(x) = e^{-(x^2+1)^2}\cdot 2x - e^{-x^4}\cdot 2x$$

$$= 2x\left[e^{-(x^2+1)^2} - e^{-x^4}\right]$$

$\because (x^2+1)^2 > x^4$

$\therefore\ e^{+(x^2+1)^2} > e^{x^4}$ $\therefore e^{-(x^2+1)^2} < e^{-x^4}$

$\therefore\ e^{-(x^2+1)^2} - e^{-x^4} < 0$

$\therefore\ f'(x) \ge 0\ \forall\ x \le 0$

Therefore, $f(x)$ increases when $x \le 0$.

22. a. $\int_0^{t^2} xf(x)\,dx = \frac{2}{5}t^5$ (Here, $t > 0$)

Differentiating both sides w.r.t. t, we get

$$t^2 f(t^2) \times 2t = \frac{2}{5} \times 5t^4$$

$\therefore f(t^2) = t$

Put $t = \frac{2}{5}$. Then $f\left(\frac{4}{25}\right) = \frac{2}{5}$.

23. b. $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$$

24. c. $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]dx$ (1)

$$= \int_{-2}^{0} [(x+1)^3 + 2 + (x+1)\cos(x+1)]dx$$

$$= \int_{-2}^{0} [(-2-x+1)^3 + 2 + (-2-x+1)\cos(-2-x+1)]dx$$

$\therefore I = \int_{-2}^{0} [-(1+x)^3 + 2 - (1+x)\cos(1+x)]dx$ (2)

Adding (1) and (2)

$2I = 2\int_{-2}^{0} 2$ or $I = 4$

25. a. $\lim_{x \to \frac{\pi}{4}} \frac{\int_{2}^{\sec^2 x} f(t)dt}{x^2 - \frac{\pi^2}{16}}$ $\left(\frac{0}{0} \text{ from}\right)$

$$= \lim_{x \to \frac{\pi}{4}} \frac{f(\sec^2 x) 2 \sec x \sec x \tan x}{2x}$$

(Applying L'Hospital's Rule)

$$= \frac{2f(2)}{\pi/4} = \frac{8f(2)}{\pi}$$

26. c. $\int_0^x \sqrt{1-(f'(t))^2}\, dt = \int_0^x f(t)dt$

Differentiating w.r.t. x, we get

$\sqrt{1-(f'(x))^2} = f(x)$

$\therefore$ $1 - (f'(x))^2 = (f(x))^2$

$\therefore$ $(f'(x))^2 = 1 - (f(x))^2$

$\therefore$ $f'(x) = \pm\sqrt{1-(f(x))^2}$

$\therefore$ $\int \frac{f'(x)}{\sqrt{1-(f(x))^2}} dx = \pm \int 1\, dx$

$\therefore$ $\int \frac{dt}{\sqrt{1-t^2}} = \pm \int 1\, dx$

$\therefore$ $\sin^{-1} t = \pm(x+c)$

$\therefore$ $t = f(x) = \pm\sin(x+c)$

Now, $f(0) = 0$

$\therefore$ $c = 0$

Thus, $f(x) = \sin x$ (as $f(x)$ is non-negative function in $[0, 1]$).

Hence, correct option is (c) as $x > \sin x$ for $\forall\ x > 0$.

27. a $\int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right]_0^1 - 4[\tan^{-1} x]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$$

28. b. $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)dt}{t^4+4}$

$$= \lim_{x \to 0} \frac{\int_0^x \frac{t \ln(1+t)dt}{t^4+4}}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{x \ln(1+x)}{(x^4+4)}}{3x^2}$$

$$= \lim_{x \to 0} \frac{1}{3} \frac{\log_e(1+x)}{x(x^4+4)}$$

$$= \frac{1}{12}$$

29. b $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$ (1)

Now, $f(f^{-1}(x)) = x$

$\therefore f'(f^{-1}(x))(f^{-1}(x))' = 1$

$\therefore (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$

$f(0) = 2 \therefore f^{-1}(2) = 0$

$\therefore (f^{-1})'(2) = \frac{1}{f'(0)}$

$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$

Differentiating w.r.t. x, we get

$e^{-x}(f'(x) - f(x)) = \sqrt{x^4+1}$

Put $x = 0$

$\therefore f'(0) - 2 = 1$

or $f'(0) = 3$

$\therefore (f^{-1})'(2) = 1/3$

30. a Put $x^2 = t$ or $2x\, dx = dt$

$\therefore I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt$ (1)

$\therefore I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t} dt$ (2)

Using $\left(\int_a^b f(d)dx = \int_a^b f(a+b-x)dx \text{ and } \ln 2 + \ln 3 = \ln 6\right)$

Adding (1) and (2)

$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1\,dt$ or $I = \frac{1}{4} \ln \frac{3}{2}$.

31. c $R_1 = \int_{-1}^{2} x f(x)dx = \int_{-1}^{2} (2-1-x)f(2-1-x)\, dx$

$$= \int_{-1}^{2} (1-x)f(1-x)\, dx = \int_{-1}^{2} (1-x) f(x)dx$$

Hence, $2R_1 = \int_{-1}^{2} f(x)\,dx = R_2$.

32. b.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{x^2 + \log_e\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos x\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log_e\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} x^2\cos x\,dx + 0\left(\because \log_e\left(\frac{\pi+x}{\pi-x}\right)\cos x \text{ is an odd function}\right)$$

$$= 2[x^2\sin x + 2x\cos x - 2\sin x]_0^{\pi/2}$$

$$= 2\left[\frac{\pi^2}{4} - 2\right] = \frac{\pi^2}{2} - 4$$

33. d Given $f'(x) - 2f(x) < 0 \therefore f'(x)e^{-2x} - 2e^{-2x}f(x) < 0$

$\therefore \quad \frac{d}{dx}(f(x)e^{-2x}) < 0$

Thus, $g(x) = f(x)e^{-2x}$ is decreasing function.

Also, $f(1/2) = 1$.

For $x > \frac{1}{2}$

$g(x) < g(1/2)$ or $f(x)e^{-2x} < f(1/2)e^{-1}$ or $f(x) < e^{2x-1}$

$$\therefore 0 < \int_{1/2}^{1} f(x)\,dx < \int_{1/2}^{1} e^{2x-1}\,dx \text{ or } 0 < \int_{1/2}^{1} f(x)\,dx < \frac{e-1}{2}$$

34. b. $Fx = \int_0^{x^2} f(\sqrt{t})\,dt$

$F(0) = 0$

$F'(x) = 2x\,f(x) = f'(x)$

$\Rightarrow \quad \int \frac{f'(x)}{f(x)}dx = \int 2x\,dx$

$\Rightarrow \quad \log_e f(x) = x^2 + c$

$\Rightarrow \quad f(x) = e^{x^2+c}$

$\Rightarrow \quad f(x) = e^{x^2}$ $\quad (\because f(0) = 1)$

$\Rightarrow \quad F(x) = \int_0^{x^2} e^t\,dt$

$\Rightarrow \quad F(x) = e^{x^2} - 1$ $\quad (\because F(0) = 0)$

$\Rightarrow \quad F(2) = e^4 - 1$

35. a $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec} x)^{17}\,dx$

Let $e^u + e^{-u} = 2\operatorname{cosec} x$,

For $x = \frac{\pi}{4}, u = \ln(1+\sqrt{2})$

For $x = \frac{\pi}{2}, u = 0$

Also, $\operatorname{cosec} x + \cot x = e^u$ and $\operatorname{cosec} x - \cot x = e^{-u}$

$\Rightarrow \quad \cot x = \frac{e^u - e^{-u}}{2}$

Also $(e^u - e^{-u})\,du = -2\operatorname{cosec} x\cot x\,dx$

$$\Rightarrow \quad I = -\int_{\ln(1+\sqrt{2})}^{0} (e^u + e^{-u})^{17}\frac{(e^u - e^{-u})}{2\operatorname{cosec} x\cot x}du$$

$$= -2\int_{\ln(1+\sqrt{2})}^{0} (e^u + e^{-u})^{16}\,du$$

$$= \int_0^{\ln(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\,du$$

36. d. $f'(x) = \frac{192x^3}{2+\sin^4(\pi x)}\ \forall x \in R;\ f\left(\frac{1}{2}\right) = 0$

Now, $64x^3 \le f'(x) \le 96x^3\ \forall x \in \left[\frac{1}{2}, 1\right]$

$$\int_{1/2}^{x} 64x^3\,dx \le \int_{1/2}^{x} f'(x)\,dx \le \int_{1/2}^{x} 96x^3\,dx$$

So, $\quad 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}\ \forall x \in \left[\frac{1}{2}, 1\right]$

$$\int_{1/2}^{1} (16x^4 - 1)\,dx \le \int_{1/2}^{1} f(x)\,dx \le \int_{1/2}^{1}\left(24x^4 - \frac{3}{2}\right)dx$$

$$\frac{16}{5}\cdot\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^{1} f(x)\,dx \le \frac{24}{5}\cdot\frac{31}{32} - \frac{3}{4}$$

$$\Rightarrow \quad \frac{26}{10} \le \int_{1/2}^{1} f(x)\,dx \le \frac{78}{20}$$

Multiple Correct Answers Type

1. a. $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$

Differentiating both sides w.r.t. x, we get

$f(x) = 1 + 0 - xf(x)$

$\therefore (x+1)f(x) = 1$

$\therefore f(x) = \frac{1}{x+1}$

or $f(1) = \frac{1}{2}$

2. a. $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x - [x])\,dx$

$$= \int_{-1}^{1} x\,dx - \int_{-1}^{1}[x]\,dx$$

$$= 0 - \int_{-1}^{1}[x]dx \quad (1)$$

$[\because x$ is an odd function$]$

$$= -\int_{-1}^{0}(-1)dx - \int_0^1 0\,dx$$

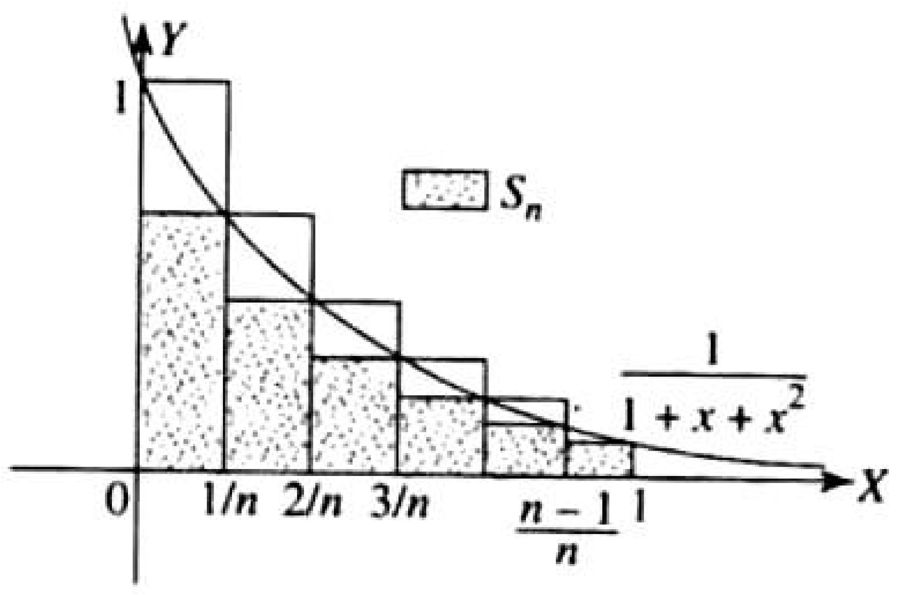
$= 1$

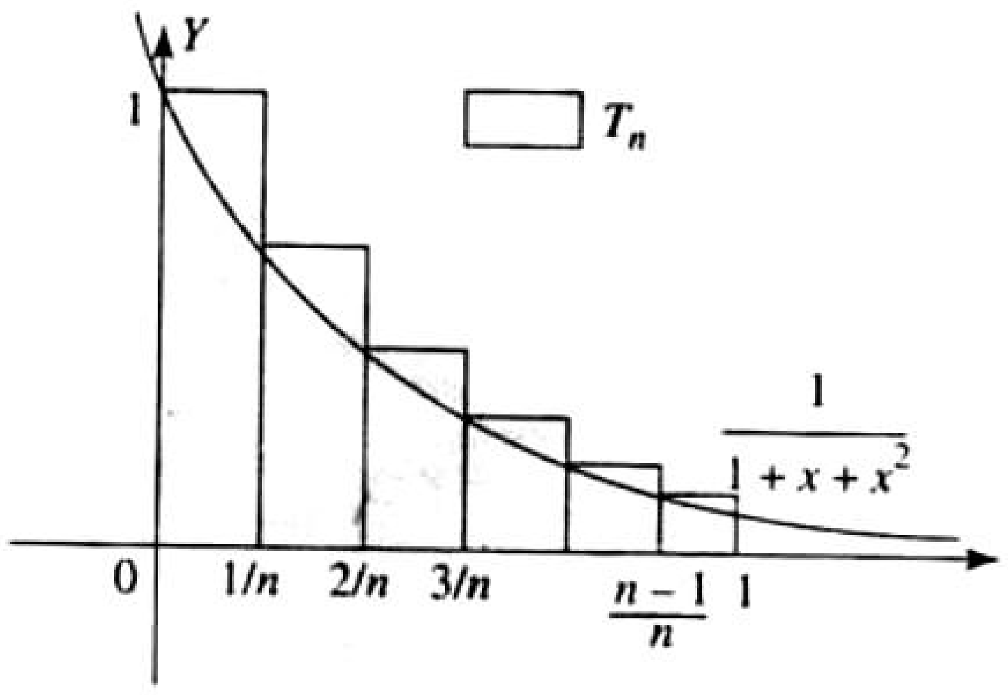
3. a., d. $$S_n=\sum_{k=1}^{n}\frac{n}{n^2+kn+k^2}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)+\left(\frac{k}{n}\right)^2}$$

and $$T_n=\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{1+\left(\frac{k}{n}\right)+\left(\frac{k}{n}\right)^2}$$

So, consider function $f(x)=\frac{1}{1+x+x^2}$





From the graph $S_n<\int_0^1\frac{1}{1+x+x^2}dx<T_n$

Now, $\int_0^1\frac{dx}{1+x+x^2}$

$$=\int_0^1\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$=\frac{\pi}{3\sqrt{3}}$$

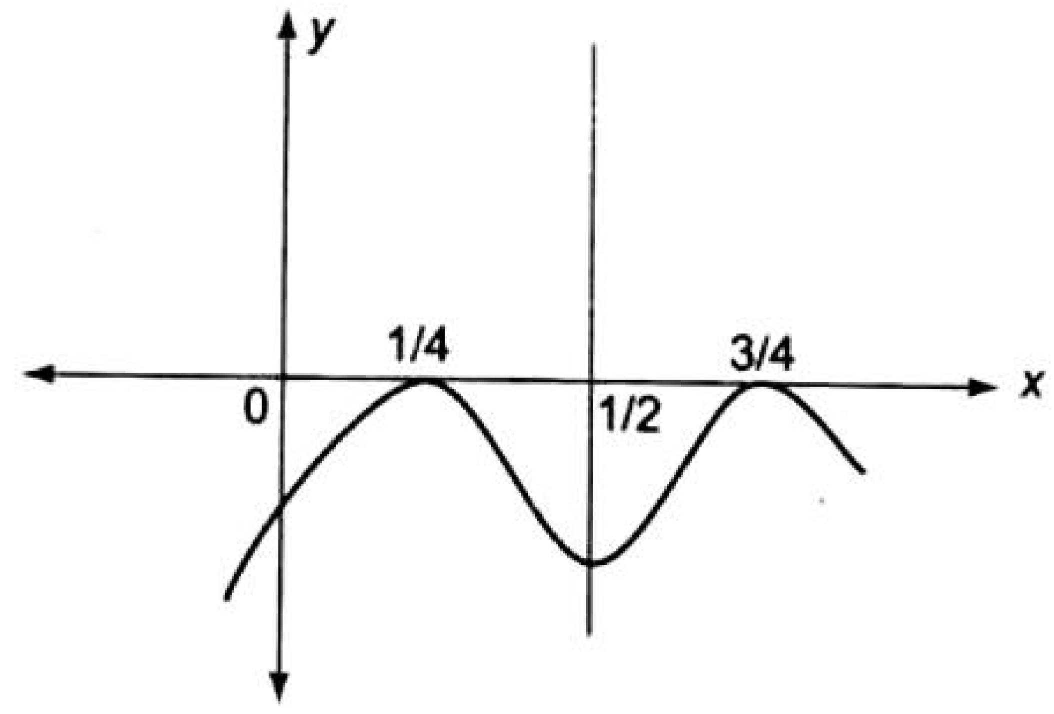
$\therefore\quad S_n<\frac{\pi}{3\sqrt{3}}<T_n$

4. a., b., c., d.

$f(x)=f(1-x)$

Replacing x by $\frac{1}{2}+x$, we get

$$f\left(\frac{1}{2}+x\right)=f\left(\frac{1}{2}-x\right) \qquad (1)$$



$$\int_{-1/2}^{1/2}f\left(x+\frac{1}{2}\right)\sin x\,dx=\int_{-1/2}^{1/2}f\left(-x+\frac{1}{2}\right)\sin(-x)dx$$

$$=-\int_{-1/2}^{1/2}f\left(x+\frac{1}{2}\right)\cdot\sin x\,dx$$

$$\therefore\int_{-1/2}^{1/2}f\left(x+\frac{1}{2}\right)\sin x\,dx=0$$

Also, $f'(x)=-f'(1-x)$ (2)
and for $x=1/2$, we have $f'(1/2)=0$.

Also, $\int_{1/2}^{1}f(1-t)e^{\sin\pi t}\,dt=-\int_{1/2}^{0}f(y)\,e^{\sin\pi y}\,dy$

(by putting, $1-t=y$)

Since $f'(1/4)=0, f'(3/4)=0$. [from equation (2)]

Also, $f'(1/2)=0$. [from equation (2)]

Thus, $f'(x)=0$ at least twice in [0, 1]. (by Rolle's theorem)

5. a., b., c.

$$I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx \qquad (1)$$

$$=\int_{-\pi}^{\pi}\frac{\sin(-nx)}{(1+\pi^{-x})\sin(-x)}dx$$

$$\therefore\ I_n=\int_{-\pi}^{\pi}\frac{\pi^x\sin nx}{(\pi^x+1)\sin x}dx \qquad (2)$$

Adding (1) and (2)

$$2I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx$$

$$=2\int_0^{\pi}\frac{\sin nx}{\sin x}dx \qquad \left(\because \frac{\sin nx}{\sin x}\text{ is an even function}\right)$$

Now, $I_{n+2}-I_n=\int_0^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx$

$$=\int_0^{\pi}\frac{2\cos(n+1)x\sin x}{\sin x}dx=0$$

$\Rightarrow\quad I_{n+2}=I_n$

$\therefore I_1=\pi, I_2=\int_0^{\pi}2\cos x\,dx=0$

$\therefore I_1=I_3=I_5=\ldots=\pi$
and $I_2=I_4=I_6=\ldots=0$

6. b., c. $f(x)=\ln x+\int_0^x \sqrt{1+\sin t}\,dt$

$$\Rightarrow \quad f'(x)=\frac{1}{x}+\sqrt{1+\sin x}$$

$$\Rightarrow \quad f''(x)=-\frac{1}{x^2}+\frac{\cos x}{2\sqrt{1+\sin x}}$$

$f''(x)$ is not defined at for $x=2n\pi-\frac{\pi}{2}, n\in N$.
So, option (a) is wrong.
$f'(x)$ exists for all $x>0$. $(\because\ 1+\sin x\ge 0)$

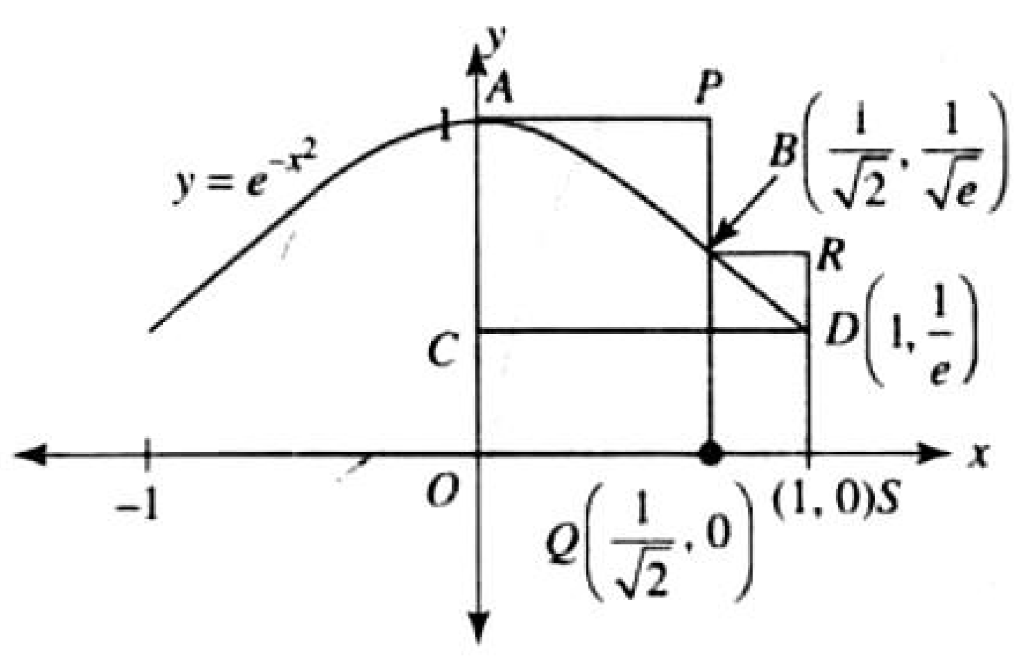
Also $f'(x)$ is continuous and differentiable as $\frac{1}{x}$ and $\sqrt{1+\sin x}$ are continuous and differentiable.

For $x>1$, $f(x)=\ln x+\int_0^x \sqrt{1+\sin t}\,dt>0$

Also, $f'(x)=\frac{1}{x}+\sqrt{1+\sin x}>0$

Further, $\frac{1}{x}+\sqrt{1+\sin x}$ is bounded and $\ln x+\int_0^x\sqrt{1+\sin t}\,dt$ is unbounded for $x\to\infty$.
Thus, there exists $\alpha>1$ such that $|f'(x)|<|f(x)|$ for $x\in(\alpha,\infty)$.
$|f(x)|+|f'(x)|\le\beta$ is wrong as $f(x)$ is monotonically increasing and it is unbounded, while β is finite.

7. a., b., d.



$S>\frac{1}{e}$ (As area of rectangle $OCDS=1/e$)

Since $e^{-x^2}\ge e^{-x}\ \forall\, x\in[0,1]$, we have

$$S>\int_0^1 e^{-x}\,dx=\left(1-\frac{1}{e}\right)$$

Area of rectangle $OAPQ$ + Area of rectangle $QBRS>S$

$$\text{or } S<\frac{1}{\sqrt{2}}(1)+\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

$$\text{Since } \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)<1-\frac{1}{e}$$

Option (c) is incorrect.

8. a., b., c., d.

$$f(x)=\int_0^x e^{t^2}\cdot(t-2)(t-3)dt$$

$$\therefore f'(x)=e^{x^2}(x-2)(x-3)$$

0 + 2 – 3 +

Clearly, maxima at $x=2$, minima at $x=3$ and decreasing in $x\in(2,3)$.

$f'(x)=0$ for $x=2$ and $x=3$ (Rolle's theorem)
So, there exist $c\in(2,3)$ for which $f''(c)=0$.

9. b., d. $\lim\limits_{n\to\infty}\dfrac{(1^a+2^a+\ldots+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]}$

$$=\frac{\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{a-1}\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\left(a+\frac{r}{n}\right)}$$

$$=\frac{\int_0^1 x^a\,dx}{\int_0^1 (a+x)dx}\qquad\left(\because\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{a-1}=1^{a-1}=1\right)$$

$$=\frac{2}{(2a+1)(a+1)}=\frac{2}{120}$$

$\therefore a=7$ or $-\frac{17}{2}$ (Given)

10. a., c.
For continuity at $x=a$

$$\lim_{x\to a^-}g(x)=0$$

$$g(a)=\int_a^a f(t)dt=0$$

$$\lim_{x\to a^+}g(x)=\lim_{x\to a^+}\int_a^x f(t)dt=0$$

Hence, $g(x)$ is continuous at $x=a$.
For continuity at $x=b$

$$\lim_{x\to b^-}g(x)=\lim_{x\to b^-}\int_a^x f(t)dt=\int_a^b f(t)dt$$
$$=\lim_{x\to b^+}g(x)=g(b)$$

Thus, $f(x)$ is continuous at $x=b$.

$$g'(x)=\begin{cases}0, & x<a\\ f(x), & a<x<b\\ 0, & x>b\end{cases}$$

Since $f(x)\ge 1$ for $x\in[a,b]$, $g(x)$ is non-differentiable at $x=a$ and $x=b$.

11. a., c., d.

$$f(x)=\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$$

$$\Rightarrow \quad f'(x)=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}-\frac{e^{-\left(\frac{1}{x}+x\right)}}{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

$$=\frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}>0 \text{ for } x\in[1,\infty)$$

Therefore, $f(x)$ is increasing in $[1,\infty)$.
$f'(x)>0$ for $x\in(0,1)$.
Hence, $f(x)$ is increasing.

Also, $f(x)+f\left(\frac{1}{x}\right)$

$$=\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}+\int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$$

$$=0$$

$$g(x)=f(2^x)=\int_{2^{-x}}^{2^x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$$

$$\therefore \quad g(-x)=\int_{2^x}^{2^{-x}}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt=-g(x)$$

Hence, $f(2^x)$ is an odd function.

12. a., c.

Let $\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt=I_1$

$$I_2=\int_{\pi}^{2\pi} e^t(\sin^6 at+\cos^4 at)dt$$

Put $t=x+\pi$

$\therefore \quad dt=dx$

For $a=2$ and $a=4$

$$\therefore \quad I_2=\int_0^{\pi} e^{x+\pi}(\sin^6 ax+\cos^4 ax)dt$$

$$=e^{\pi}I_1$$

Similarly,

$$\int_{2\pi}^{3\pi} e^t(\sin^6 at+\cos^4 at)dt=e^{2\pi}I_1$$

and $$\int_{3\pi}^{4\pi} e^t(\sin^6 at+\cos^4 at)dt=e^{3\pi}I_1$$

$$\therefore \quad \frac{\int_0^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt}$$

$$=\frac{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt+\int_{\pi}^{2\pi} e^t(\sin^6 at+\cos^4 at)dt+\int_{2\pi}^{3\pi} e^t(\sin^6 at+\cos^4 at)dt+\int_{3\pi}^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt}$$

$$=1+e^{\pi}+e^{2\pi}+e^{3\pi}$$

$$=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

13. a., b.

$$f(x)=(7\tan^6 x-3\tan^2 x)\cdot\sec^2 x$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} f(x)dx=\int_0^1(7t^6-3t^2)dt=(t^7-t^3)_0^1=0$$

Now, $\int_0^{\frac{\pi}{4}} xf(x)dx=\int_0^1(7t^6-3t^2)\tan^{-1}t\,dt$

$$=(\tan^{-1}t\cdot(t^7-t^3))_0^1-\int_0^1(t^7-t^3)\frac{1}{1+t^2}dt$$

$$=\int_0^1\frac{t^3(1-t^4)}{1+t^2}dt$$

$$=\int_0^1 t^3(1-t^2)dt$$

$$=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

Linked Comprehension Type

1. a. $\int_0^{\pi/2}\sin x\,dx=\frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0+\sin\frac{\pi}{2}+2\sin\frac{\pi}{4}\right)$

$$=\frac{\pi}{8}\left(1+\sqrt{2}\right)$$

2. d. $\lim_{x\to a}\frac{\int_a^x f(x)\,dx-\left(\frac{x-a}{2}\right)\left(f(x)+f(a)\right)}{(x-a)^3}=0$

$$\lim_{h\to 0}\frac{\int_a^{a+h} f(x)\,dx-\frac{h}{2}\left(f(a+h)+f(a)\right)}{h^3}=0$$

$$\therefore \lim_{h\to 0}\frac{f(a+h)-\frac{1}{2}[f(a)+f(a+h)]-\frac{h}{2}(f'(a+h))}{3h^2}=0$$

[Using L' Hospital's rule]

$$\therefore \lim_{h\to 0}\frac{\frac{1}{2}f(a+h)-\frac{1}{2}f(a)-\frac{h}{2}f'(a+h)}{3h^2}=0$$

$$\therefore \lim_{h\to 0}\frac{\frac{1}{2}f'(a+h)-\frac{1}{2}f'(a+h)-\frac{h}{2}f''(a+h)}{6h}=0$$

[Using L' Hospital's rule]

$$\therefore \lim_{h\to 0}\frac{-f''(a+h)}{12}=0$$

$\therefore f''(a)=0\ \forall\ a\in R$

Thus, $f(x)$ must be of maximum degree 1.

3. b. $f''(x)<0\ \forall\ x\in(a,b)$, for $c\in(a,b)$

$$F(c)=\frac{c-a}{2}\left(f(a)+f(c)\right)+\frac{b-c}{2}\left(f(b)+f(c)\right)$$

$$=\frac{b-a}{2}f(c)+\frac{c-a}{2}f(a)+\frac{b-c}{2}f(b)$$

or $F'(c) = \frac{b-a}{2} f'(c) + \frac{1}{2} f(a) - \frac{1}{2} f(b)$

$$= \frac{1}{2} [(b-a) f'(c) + f(a) - f(b)]$$

$\therefore F''(c) = \frac{1}{2}(b-a) f''(c) < 0$

$[\because f''(x) < 0 \ \forall x \in (a, b) \text{ and } b > a]$

Therefore, $F(c)$ is maximum at the point $(c, f(c))$ where

$$F'(c) = 0 \text{ or } f'(c) = 2\left(\frac{f(b) - f(a)}{b-a}\right).$$

4. b. $f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$; $0 < a < 2$

$$g'(x) = \frac{f'(e^x)e^x}{1+e^{2x}}$$

$$= \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})}$$

Now, $g'(x) > 0$ if $e^{2x} - 1 > 0$ or $e^{2x} > 1$ or $x > 0$

And, $g'(x) < 0$ if $e^{2x} - 1 < 0$ or $e^{2x} < 1$ or $x < 0$

5. b. $g'(x) = \left(\frac{2(x-1)}{x+1} - \log_e x\right) f(x).$

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$$

$$= 1 - \frac{2ax}{x^2 + ax + 1}$$

$$= 1 - \frac{2a}{\left(x + \frac{1}{x}\right) + a}$$

For $x > 1$, $x + \frac{1}{x} > 2$

$\therefore \frac{2a}{\left(x + \frac{1}{x}\right) + a} < 1$ $\quad (\therefore 0 < a < 2)$

$\therefore f(x) > 0$

Let $h(x) = \left(\frac{2(x-1)}{x+1} - \log_e x\right)$

$\Rightarrow \quad h'(x) = \left(\frac{4}{(x+1)^2} - \frac{1}{x}\right) = \frac{-(x-1)^2}{(x+1)^2} < 0$

Also, $h(1) = 0$ so, $h(x) < 0$. $\therefore g'(x) < 0 \quad \forall x > 1$

Therefore, $g(x)$ is decreasing on $(1, \infty)$.

6. a.

$$g\left(\frac{1}{2}\right) = \lim_{h \to 0^+} \int_h^{1-h} t^{-1/2}(1-t)^{-1/2} dt$$

$$= \int_0^1 \frac{dt}{\sqrt{t - t^2}}$$

$$= \int_0^1 \frac{dt}{\sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_0^1$$

$$= \sin^{-1} 1 - \sin^{-1}(-1) = \pi$$

7. d. $g(a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$

$\Rightarrow \quad g(1-a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1} dt$

$$= \lim_{h \to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a} dt$$

$$= \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1}(1-(1-t))^{-a} dt$$

$$= \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1} t^{-a} dt$$

Thus, $g(a) = g(1-a)$

$\Rightarrow \quad g'(a) = -g'(1-a)$

$\Rightarrow \quad g'(1/2) = -g'(1 - 1/2)$

$\Rightarrow \quad g'(1/2) = 0$

8. a., b., c.

$f(x) = x F(x)$ (1)

$\therefore \quad f'(x) = xF'(x) + F(x)$ (2)

$\Rightarrow \quad f'(1) = F'(1) + F(1) = F'(1) < 0$

$F(1) = 0$ and $F(3) = -4$

Also, $F'(x) < 0$ for all $x \in (1/2, 3)$.

So, $F(x)$ is decreasing and hence $F(2) < 0$.

$\therefore \quad f(2) = 2F(2) < 0$

Also, for $x \in (1, 3)$,

$f'(x) = xF'(x) + F(x) < 0$

9. c., d.

$$\int_1^3 x^3 F''(x) dx = 40$$

$\Rightarrow \quad [x^3 F'(x)]_1^3 - \int_1^3 3x^2 F'(x) dx = 40$

$\Rightarrow \quad [x^2 f'(x) - xf(x)]_1^3 - 3(-12) = 40$ (Using (1) and (2))

$\Rightarrow \quad 9f'(3) - 3f(3) - f'(1) + f(1) = 4$

$\Rightarrow \quad 9f'(3) + 36 - f'(1) + 0 = 4$ $\quad (\because F(1), \therefore f(1) = 0)$

$\Rightarrow \quad 9f'(3) - f'(1) + 32 = 0$

$$\int_1^3 f(x) dx$$

$$= \int_1^3 xF(x) dx$$

$$= \left[\frac{x^2}{2} F(x)\right]_1^3 - \frac{1}{2}\int_1^3 x^2 F'(x) dx$$

$$= \frac{9}{2} F(3) - \frac{1}{2} F(1) + 6$$

$$= -18 + 6 = -12$$

Matching Column Type

1. (iii) – (b), (c)

$$\left|\int_0^1 (1-y^2)dy\right| + \left|\int_1^0 (y^2-1)dy\right|$$

$$= 2\int_0^1 (1-y^2)dy = \frac{4}{3}$$

Also, $\left|\int_0^1 \sqrt{1-x}\,dx\right| + \left|\int_{-1}^0 \sqrt{1+x}\,dx\right|$

$$= 2\int_0^1 \sqrt{1-x}\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx$$

$$= 2\cdot\frac{2}{3}\cdot x^{3/2}\Big|_0^1 = \frac{4}{3}$$

Note: Solutions of the remaining parts are given in their respective chapters.

2. (i)-(a)

$$I = \int_0^{\pi/2} (\sin x)^{\cos x}(\cos x\cdot\cot x - \log(\sin x)^{\sin x})dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{d}{dx}(\sin x)^{\cos x}dx = [(\sin x)^{\cos x}]_0^{\pi/2} = 1$$

Note: Solutions of the remaining parts are given in their respective chapters.

3. (a) – (s); (b) – (s); (c) – (p); (d) – (r).

a. $\int_{-1}^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$$

b. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1}x\right)_0^1 = \sin^{-1}(1) - \sin^{-1}(0)$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

c. $\int_2^3 \frac{dx}{1-x^2} = \left[\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right]_2^3 = \frac{1}{2}[\log 2 - \log 3]$

$$= \frac{1}{2}\log 2/3$$

d. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1}x\right]_1^2 = \sec^{-1}2 - \sec^{-1}1 = \frac{\pi}{3}$

4. (b) – (p), (t)

$$I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$$

Putting $x = t + 3$

$$I = \int_{-2}^2 (t+2)\ (t+1)\ t\ (t-1)\ (t-2)dt$$

$$= \int_{-2}^2 t(t^2-1)(t^2+1)dt$$

$= 0$ (as $t(t^2-1)(t^2+1)$ is an odd function)

Note: Solutions of the remaining parts are given in their respective chapters.

5. (b) – (p)

$$\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$$

$$\int_a^b f(x)\,dx = \frac{3}{2}(b^2-a^2) + a^2 - b^2 = \left(\frac{b^2-a^2}{2}\right)$$

$$\Rightarrow \quad f(x) = x \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

(c) – (s)

$$\frac{\pi^2}{\log_e 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$$

$$= \frac{\pi^2}{\log_e 3}\left(\frac{\log_e|(\sec\pi x + \tan\pi x)|_{7/6}^{5/6}}{\pi}\right)$$

$$= \frac{\pi}{\log_e 3}\left(\log_e\left|\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}\right| - \log_e\left|\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}\right|\right)$$

$$= \pi$$

Note: Solutions of the remaining parts are given in their respective chapters.

6. d.

(p) Let $f(x) = ax^2 + bx$, $\quad (\because\ f(0) = 0)$

Given $\int_0^1 f(x)dx = 1$

$\Rightarrow \quad 2a + 3b = 6$

$\Rightarrow \quad (a, b) \equiv (0, 2)$ and $(3, 0)$

(q) $f(x) = \sin(x^2) + \cos(x^2)$

$$= \sqrt{2}\cos\left(x^2 - \frac{\pi}{4}\right)$$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi,\ n \in Z$

$$\Rightarrow \quad x^2 = 2n\pi + \frac{\pi}{4},\ n \in Z$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{x}{4}}, \pm\sqrt{\frac{9\pi}{4}} \text{ as } x \in [-\sqrt{13}, \sqrt{13}]$$

(r) $I = \int_{-2}^2 \frac{3x^2}{(1+e^x)}dx \quad (1)$

$$= \int_{-2}^2 \frac{3(-x)^2}{1+e^{-x}}dx$$

$\therefore\ I = \int_{-2}^2 \frac{e^x(3x^2)}{e^x+1}dx \quad (2)$

Adding (1) and (2)

$$\Rightarrow \quad I + I = \int_{-2}^2 \frac{3x^2}{(1+e^x)}dx + \int_{-2}^2 \frac{e^x(3x^2)}{e^x+1}dx$$

$$= \int_{-2}^2 3x^2\,dx$$

$$= 2\int_0^2 3x^2\,dx$$

$$= 16$$

$\Rightarrow \quad I = 8$

(s) We have $I = \dfrac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}{\int_{0}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}$

Let, $f(x) = \cos 2x \ln\left(\frac{1+x}{1-x}\right)$

$\therefore \; f(-x) = \cos(-2x)\ln\left(\frac{1-x}{1+x}\right)$

$= -\cos(2x)\ln\left(\frac{1+x}{1-x}\right)$

$= -f(x)$

Thus, $f(x)$ is an odd function.

$\Rightarrow \quad I = 0$

Integer Answer Type

1. (0) $f(x) = \int_0^x f(t)\,dt$, i.e., $f(0) = 0$

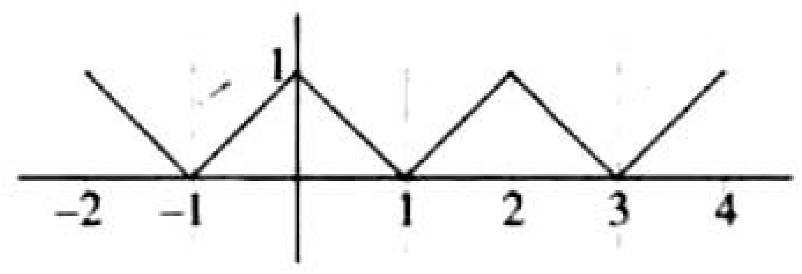
Also, $f'(x) = f(x), x > 0$

or $f(x) = ke^x, x > 0$

Since $f(0) = 0$ and $f(x)$ is continuous, $f(x) = 0 \;\forall\; x > 0$

$\therefore f(\ln 5) = 0$

2. (4) $f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ 1-x, & 0 \le x < 1 \end{cases}$



From the graph $f(x)$ is periodic with period 2 and also even function.

Also $\cos \pi x$ has period 2

$\therefore I = \int_{-10}^{10} f(x) \cos \pi x\, dx$

$= 2\int_0^{10} f(x)\cos \pi x\, dx \qquad (\because f(x) \cos \pi x$ is an even function)

$= 2 \times 5 \int_0^2 f(x) \cos \pi x\, dx$

$= 10\left[\int_0^1 (1-x)\cos \pi x\, dx + \int_1^2 (x-1)\cos \pi x\, dx\right] = 10(I_1 + I_2)$

$I_2 = \int_1^2 (x-1)\cos \pi x\, dx \qquad$ (put $x - 1 = t$)

$I_2 = -\int_0^1 t\cos \pi t\, dt$

$I_1 = \int_0^1 (1-x)\cos \pi x\, dx = -\int_0^1 x\cos(\pi x) dx$

$\therefore I = 10\left[-2\int_0^1 x\cos \pi x\, dx\right]$

$= -20\left[x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$

$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2}$

$\therefore \frac{\pi^2}{10} I = 4$

3. (2) $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$

$= \left[4x^3 \frac{d}{dx}(1-x^2)^5\right]_0^1 - \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5\, dx$

(Integrating using by parts)

$= [4x^3 \times 5(1-x^2)^4(-2x)]_0^1$

$-12\left[[x^2(1-x^2)^5]_0^1 - \int_0^1 2x(1-x^2)^5\, dx\right]$

$= 0 - 0 - 12\,[0-0] + 12 \int_0^1 2x(1-x^2)^5\, dx$

$= 12\left[-\frac{(1-x^2)^6}{6}\right]_0^1$

$= 12\left[0 + \frac{1}{6}\right] = 2$

4. (7) $f(x)$ is continuous odd function and vanishes exactly at one point.

$\therefore \quad f(0) = 0$

$F(x) = \int_{-1}^{x} f(t)dt$

$= \int_{-1}^{1} f(t)dt + \int_{1}^{x} f(t)dt$

$= 0 + \int_1^x f(t)dt$ (as $f(t)$ is an odd function)

$f(t)$ is odd function

$\therefore \quad f(f(t))$ is also odd function

$\therefore \quad |f(f(t))|$ is an even function

$\therefore \quad t|f(f(t))|$ is an odd function

$\therefore \quad G(x) = \int_{-1}^{x} t\,|f(f(t))|dt = \int_1^x t\,|f(f(t))|dt$

Now, $\lim_{x\to 1} \frac{F(x)}{G(x)} \qquad \left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 1} \frac{f(x)}{x|f(f(x))|} \qquad$ (Using L'Hospital's Rule)

$= \dfrac{\frac{1}{2}}{1\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$ (Given)

$\Rightarrow \quad f\left(\frac{1}{2}\right) = 7$

5. (9) $\alpha = \int_0^1 e^{9x+3\tan^{-1}x}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t$

$\Rightarrow \left(9+\frac{3}{1+x^2}\right)dx = dt$

$\Rightarrow \left(\frac{12+9x^2}{1+x^2}\right)dx = dt$

$\Rightarrow \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$

$\Rightarrow \log_e(1+\alpha) = 9 + \frac{3\pi}{4}$

6. (3) $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\, dt$

$\therefore F'(x) = 2\left(\cos\left(x^2+\frac{\pi}{6}\right)\right)^2 2x - 2\cos^2 x$

According to the question,

$\therefore \quad F'(a) + 2 = \int_0^a f(x)dx$

$\Rightarrow \quad 2\left(\cos\left(a^2+\frac{\pi}{6}\right)\right)^2 2a - 2\cos^2 a + 2 = \int_0^a f(x)dx$

Differentiating w.r.t. a, we get

$4\cos^2\left(a^2+\frac{\pi}{6}\right) + 4a\times 2\cos\left(a^2+\frac{\pi}{6}\right)\left(-\sin\left(a^2+\frac{\pi}{6}\right)\right)$
$\times 2a + 4\cos a \sin a = f(a)$

$\therefore \quad f(0) = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$

7. (0) $I = \int_{-1}^{2} \frac{x[x^2]}{2+[x+1]}dx$

$= \int_{-1}^{2} \frac{x[x^2]}{3+[x]}dx$

$= \int_{-1}^{0} \frac{0}{3-1}dx + \int_0^1 \frac{0}{3+0}dx + \int_1^{\sqrt{2}} \frac{x\cdot 1}{3+1}dx + 0$

$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}}$

$= \frac{2-1}{8} = \frac{1}{4}$

$\therefore \quad 4I - 1 = 0$

Fill in the Blanks Type

1. Given that

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 \to R_1 - \sec x \cdot R_3$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1, we get

$f(x) = (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(\cos^4 x - \cos^2 x)$

$= \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x\right)\cos^2 x\left(\cos^2 x - 1\right)$

$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\cos^2 x \sin^2 x}\right]\cos^2 x \sin^2 x$

$= -\sin^2 x - \cos^3 x(1 - \sin^2 x)$

$= -\sin^2 x - \cos^5 x$

$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2}\left(\sin^2 x + \cos^5 x\right)dx$

$= -\int_0^{\pi/2}\left[\frac{1-\cos 2x}{2} + \cos x(1-\sin^2 x)^2\right]dx$

$= -\left[\frac{x+\frac{\sin 2x}{2}}{2}\right]_0^{\frac{\pi}{2}} - \left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right)_1^1$, where $t = \sin x$

$= -\frac{\pi}{4} - \left(1 - \frac{2}{3} + \frac{1}{5}\right)$

$= -\left(\frac{15\pi+32}{60}\right)$

2. When $x = 0$, $x^2 = 0$, and when $x = 1.5$, $x^2 = 2.25$.

Thus, $[x^2]$ is discontinuous when $x^2 = 1$ and $x^2 = 2$ or $x = 1$ and $x = \sqrt{2}$.

$\therefore \int_0^{1.5}[x^2]\,dx = \int_0^1[x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{1.5}[x^2]dx$

$= 0 + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$

$= 1(\sqrt{2}-1) + 2(1.5-\sqrt{2}) = (2-\sqrt{2})$

3. Let $I = \int_{-2}^{2}|1-x^2|\,dx = 2\int_0^2|1-x^2|\,dx$

$= 2\int_0^1\left(1-x^2\right)dx + 2\int_1^2\left(x^2-1\right)dx$

$= 2\left[x - \frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^3}{3} - x\right]_1^2$

$= 2\left[1-\frac{1}{3}\right] + 2\left[\frac{8}{3} - 2 - \frac{1}{3} + 1\right]$

$= \frac{4}{3} + \frac{8}{3} = 4$

4. $I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}\,d\phi$ (1)

$= \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin(\pi-\phi)}\,d\phi$

$$\left[\text{Using } \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx\right]$$

$$\therefore I = \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi}d\phi \qquad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi}d\phi$$

$$= \pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{1-\sin^2\phi}d\phi$$

$$= \pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi}d\phi$$

$$= \pi\int_{\pi/4}^{3\pi/4} (\sec^2\phi - \sec\phi\tan\phi)d\phi$$

$$= \pi[\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4}$$

$$= \pi[\tan 3\pi/4 - \sec 3\pi/4 - \tan\pi/4 + \sec\pi/4]$$

$$= \pi[-1+\sqrt{2}-1+\sqrt{2}]$$

$$= 2\pi(\sqrt{2}-1)$$

$$\therefore I = \pi(\sqrt{2}-1)$$

5. Let $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx \qquad (1)$

$$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)}+\sqrt{5-x}}dx$$

$$\left[\text{Using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$\therefore I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx \qquad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

or $I = \frac{1}{2}\int_2^3 1\,dx = \frac{1}{2}(3-2) = \frac{1}{2}$

6. Let us first find the functions satisfying

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5. \qquad (1)$$

Replacing x by $\frac{1}{x}$, we have $af\left(\frac{1}{x}\right) + bf(x) = x - 5. \qquad (2)$

Eliminating $f\left(\frac{1}{x}\right)$ from equations (1) and (2), we get

$$\int_1^2 f(x)\,dx = \int_1^2 \frac{\frac{a}{x} - 5a - bx + 5b}{a^2-b^2}dx$$

$$= \frac{1}{a^2-b^2}\left[a\log x - b\frac{x^2}{2} + 5(b-a)x\right]_1^2$$

$$= \frac{1}{a^2-b^2}\left[a\log 2 - 2b + 10(b-a) + \frac{b}{2} - 5(b-a)\right]$$

$$= \frac{1}{a^2-b^2}\left[a\log 2 - 5a + \frac{7b}{2}\right]$$

7. $I = \int_0^{2\pi} \frac{x\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx \qquad (1)$

$$= 2\int_0^{2\pi} \frac{(2\pi - x)\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx \qquad (2)$$

$$\left[\text{Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding equations (1) and (2), we get

$$2I = \int_0^{2\pi} \frac{2\pi\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx = 4\pi\int_0^{\pi} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx$$

$$\left[\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

$$= 8\pi\int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx \qquad (3)$$

[Using the above property again]

$$= 8\pi\int_0^{\pi/2} \frac{\cos^{2n}\left(\frac{\pi}{2}-x\right)}{\cos^{2n}\left(\frac{\pi}{2}-x\right)+\sin^{2n}\left(\frac{\pi}{2}-x\right)}dx$$

$$\left[\text{Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= 8\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \qquad (4)$$

Adding equations (3) and (4), we have

$$4I = 8\pi\int_0^{\pi/2} 1dx$$

or $I = \pi^2$

8. Let $I = \int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x}dx$

Let $\pi\ln x = t$

$$\therefore \frac{\pi}{x}dx = dt$$

$$\therefore I = \int_0^{37\pi}\sin t\,dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$$

$$= -(-1) + 1 = 2$$

9. $I = \int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k) - F(1)$

Put $x^2 = t$ $\therefore$ $2x\,dx = dt$

$$\therefore I = \int_1^{16}\frac{e^{\sin t}}{t}dt = F[(t)]_1^{16}$$

$\therefore I = F(16) - F(1)$

$\therefore k = 16$

True/False Type

1. Let $I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx$ (1)

$= \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)} dx$ (2)

[Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

Adding equations (1) and (2), we get

$2I = \int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)} dx$

$= \int_0^{2a} 1\,dx$

$= [x]_0^{2a} = 2a$ or $I = a$

Therefore, the given statement is true.

Subjective Type

1. $L = \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\left(\frac{1}{1+r/n}\right)$

Now, Lower limit $= \lim_{n\to\infty}(r/n)_{r=1} = \lim_{n\to\infty}(1/n) = 0$

Upper limit $= \lim_{n\to\infty}(r/n)_{r=5n} = \lim_{n\to\infty}(5n/5) = 5$

Then $L = \int_0^5 \frac{dx}{1+x} = [\log(1+x)]_0^5 = \log 6$

2. $\int_0^1 (tx+1-x)^n\,dx$

$= \int_0^1 [(t-1)x+1]^n\,dx$

$= \left[\frac{[(t-1)x+1]^{n+1}}{(t-1)(n+1)}\right]_0^1$

$= \frac{1}{n+1}\left[\frac{t^{n+1}}{t-1} - \frac{1}{t-1}\right]$

$\therefore \int_0^1 (tx+1-x)^n\,dx = \frac{t^{n+1}-1}{(t-1)(n+1)}$ (1)

For $\int_0^1 x^k(1-x)^{n-k}\,dx = \left[{}^nC_k\,(n+1)\right]^{-1}$

$k = 0, 1, 2, \cdots, n$

Now, $[tx+(1-x)]^n$

$= \sum_{k=0}^{n} {}^nC_k\,(tx)^k(1-x)^{n-k}$ [Using binomial theorem]

$= \sum_{k=0}^{n}\left[{}^nC_k\,x^k(1-x)^{n-k}\right]t^k$

Integrating both sides from 0 to 1 w.r.t. x, we get

$\int_0^1 [tx+(1-x)]^n\,dx = \sum_{k=0}^{n} t^k\,{}^nC_k \int_0^1 x^k(1-x)^{n-k}\,dx$

$\therefore \frac{t^{n+1}-1}{(t-1)(n+1)} = \sum_{k=0}^{n} {}^nC_k t^k\left\{\int_0^1 x^k(1-x)^{n-k}\,dx\right\}$

[Using equation (1)]

$\therefore \sum_{k=0}^{n} {}^nC_k\,t^k\left\{\int_0^1 x^k(1-x)^{n-k}\,dx\right\}$

$= \frac{1}{n+1}[1+t+t^2+t^3+\cdots+t^n]$ [Using sum of G.P.]

Equating the coefficients of t^k on both the sides, we get

${}^nC_k\int_0^1 x^k(1-x)^{n-k}\,dx = \frac{1}{n+1}$

$\therefore \int_0^1 x^k(1-x)^{n-k}\,dx = \frac{1}{{}^nC_k\,(n+1)}$

3. Let $I = \int_0^\pi x f(\sin x)\,dx$ (1)

Now, using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$I = \int_0^\pi (\pi-x) f\{\sin(\pi-x)\}\,dx$

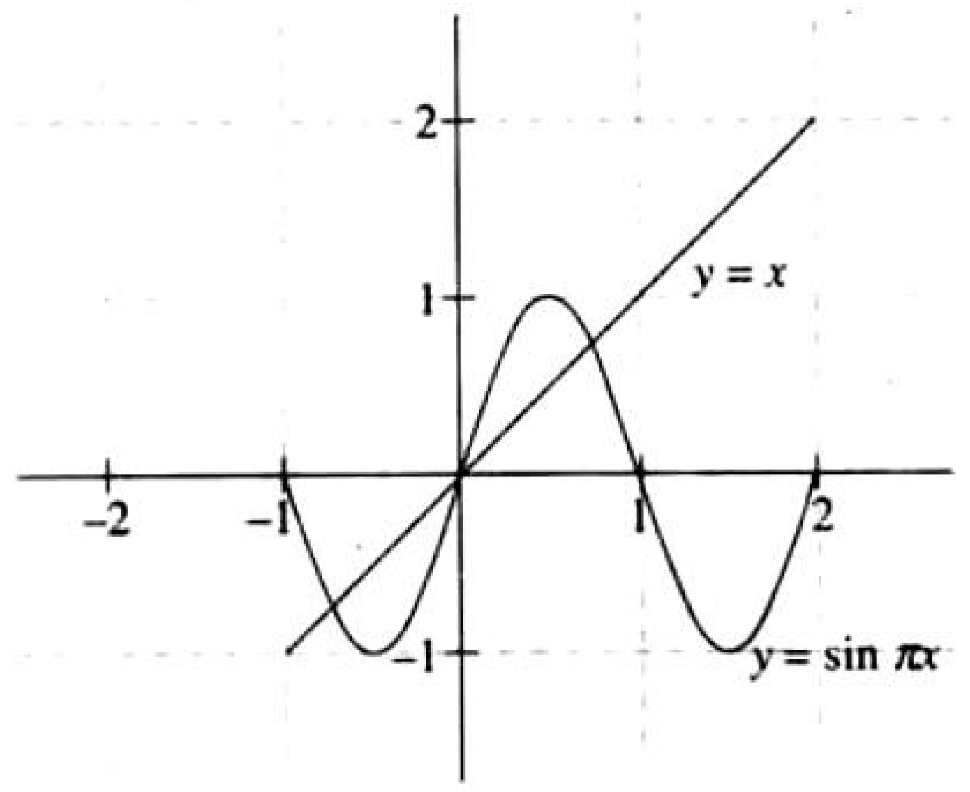
or $I = \int_0^\pi (\pi-x) f(\sin x)\,dx$ (2)

Thus, adding equations (1) and (2), we get $2I = \pi\int_0^\pi f(\sin x)\,dx$

or $I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

4. To understand the sign scheme of $y = x\sin\pi x$, lets first draw the graphs of $y = x$ and $y = \sin\pi x$

The period of $y = \sin\pi x$ is '2'.



From the graphs,

$|x\sin\pi x| = \begin{cases}(-x)(-\sin\pi x) & \text{if } -1 \le x < 0\\ x\sin\pi x & \text{if } 0 < x \le 1\\ x(-\sin\pi x) & \text{if } 1 < x \le 3/2\end{cases}$

$\therefore \int_{-1}^{3/2} |x\sin\pi x|\,dx$

$= \int_{-1}^{0} x\sin\pi x\,dx + \int_{-1}^{1} x\sin\pi x\,dx + \int_1^{3/2}(-x\sin\pi x)dx$

$= \int_{-1}^{1} x\sin\pi x\,dx - \int_1^{3/2} x\sin\pi x\,dx$

$= 2\int_0^1 x\sin\pi x\,dx - \int_1^{3/2} x\sin\pi x\,dx$

$= 2\left[\left\{x\left(\frac{-1}{\pi}\right)\cos\pi x\right\}_0^1 - \int_0^1 1\left(\frac{-1}{\pi}\right)\cos\pi x\,dx\right]$

$$-\left\{x\left(\frac{-1}{\pi}\right)\cos \pi x\right\}_1^{3/2}+\int_1^{3/2}1\left(\frac{-1}{\pi}\right)\cos\pi x\, dx$$

$$=\left(\frac{2}{\pi}\right)+\left(\frac{2}{\pi^2}\right)[\sin \pi x]_0^1+\left\{\frac{3}{(2\pi)}\right\}\cos\frac{3}{2}\pi+\left(\frac{1}{\pi}\right)$$

$$-\left(\frac{1}{\pi^2}\right)[\sin \pi x]_1^{3/2}$$

$= (2/\pi) + 0 + 0 + (1/\pi) + (1/\pi^2)$
$= (3\pi + 1)/\pi^2$

5. $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

We know that $(\sin x - \cos x)^2 = 1 - \sin 2x$

or $\sin 2x = 1 - (\sin x - \cos x)^2$

$$\therefore I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16(1-(\sin x-\cos x)^2)}dx$$

$$=\int_0^{\pi/4}\frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}dx$$

Let $\sin x - \cos x = t$

$$\therefore I=\int_{-1}^{0}\frac{dt}{25-16t^2}$$

$$=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$$

$$=\frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\log\left[\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]$$

$$=\frac{\log 9}{40}=\frac{1}{20}\log 3$$

6. Let $I=\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta \therefore dx = \cos\theta\, d\theta$
Also, when $x = 0$, $\theta = 0$, and when $x = 1/2$, $\theta = \pi/6$.

Thus, $I=\int_0^{\pi/6}\frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$

$$=\int_0^{\pi/6}\theta\sin\theta\, d\theta$$

Integrating by parts, we get

$$I=\left[\theta(-\cos\theta)\right]_0^{\pi/6}+\int_0^{\pi/6}1\cos\theta\, d\theta$$

$$=[-\theta\cos\theta+\sin\theta]_0^{\pi/6}$$

$$=\frac{-\pi}{6}\frac{\sqrt{3}}{2}+\frac{1}{2}=\frac{6-\pi\sqrt{3}}{12}$$

7. Given that $f(x)$ is integrable over any interval on real line and
$f(t+x)=f(x)$ (1)
for all real x and a real t.

Now, $\int_a^{a+t}f(x)\,dx=\int_a^0 f(x)\,dx+\int_0^t f(x)dx+\int_t^{a+t}f(x)\,dx$

In the last integral, put $x = t + y$ so that $dx = dy$.

Then $\int_t^{a+t}f(x)\,dx$

$$=\int_0^a f(t+y)\,dy=\int_0^a f(y)\,dy \qquad \text{[Using equation (1)]}$$

$$=\int_0^a f(x)\,dx$$

Hence, $\int_a^{a+t}f(x)\,dx$

$$=-\int_0^a f(x)\,dx+\int_0^t f(x)\,dx+\int_0^a f(x)\,dx$$

$=\int_0^t f(x)\,dx$, which is independent of a

Alternative Method:

$$g(x)=\int_a^{a+t}f(x)dx$$

Given that $f(x+t)=f(x)$ (1)
We have to prove that $g(x)$ is independent of 'a'.

i.e., we have to prove that $\frac{dg(x)}{da}=0$

Now, $\frac{dg(x)}{da}=f(a+t)-f(a)$

From (1), $f(a+t)=f(a)$

$\therefore \quad \frac{dg(x)}{da}=0$

Thus, $g(x)$ is independent of 'a'.

8. Let $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ (1)

$$=\int_0^{\pi/2}\frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}dx$$

[Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$]

$$\therefore I=\int_0^{\pi/2}\frac{(\pi/2-x)\sin x\cos x}{\sin^4 x+\cos^4 x}dx \qquad (2)$$

Adding equations (1) and (2), we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

or $I=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

$$=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2 x\tan x}{\tan^4 x+1}dx$$ (Dividing numerator and denominator by $\cos^4 x$)

$$=\frac{\pi}{2\times 4}\int_0^{\pi/2}\frac{2\tan x\sec^2 x\,dx}{1+\left(\tan^2 x\right)^2}$$

Put $\tan^2 x = t$ $\therefore$ $2\tan x \sec^2 x\, dx = dt$
Also, as $x \to 0, t \to 0$; as $x \to \pi/2, t \to \infty$

$$\therefore I = \frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}$$

$$= \frac{\pi}{8}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{8}[\pi/2 - 0] = \pi^2/16.$$

9. Let $I = \int_0^{\pi}\frac{x\, dx}{1+\cos\alpha \sin x}$ (1)

$$= \int_0^{\pi}\frac{(\pi - x)\, dx}{1+\cos\alpha\,(\sin(\pi - x))}$$

[using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$]

$$\therefore I = \int_0^{\pi}\frac{(\pi - x)\, dx}{1+\cos\alpha \sin x} \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi}\frac{x+\pi-x}{1+\cos\alpha \sin x}dx$$

$$= \int_0^{\pi}\frac{\pi}{1+\cos\alpha \sin x}dx$$

$$\therefore I = \frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha \sin x}dx$$

$$= \frac{\pi}{2}\times 2\int_0^{\pi/2}\frac{1}{1+\cos\alpha \sin x}dx$$

$$= \pi\int_0^{\pi/2}\frac{1}{1+\cos\alpha\times\frac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$= \pi\int_0^{\pi/2}\frac{\sec^2 x/2}{1+\tan^2 x/2 + 2\cos\alpha \tan x/2}dx$$

Put $\tan x/2 = t$ or $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

Also, when $x \to 0, t \to 0$
and when $x \to \pi/2, t \to 1$

$$\therefore I = \pi\int_0^1\frac{2dt}{t^2+(2\cos\alpha)t+1}$$

$$= 2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+1-\cos^2\alpha}$$

$$= 2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$

$$= 2\pi\cdot\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{\frac{1+\cos\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}}{1+\left(\frac{1+\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\sin\alpha}\right)}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{\sin\alpha}{1+\cos\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\tan\frac{\alpha}{2}\right)\right]$$

$$= \frac{\pi\alpha}{\sin\alpha}$$

10. $\int_0^a f(x)g(x)dx$

$$= \int_0^a f(a-x)\, g(a-x)dx$$

$$= \int_0^a f(x)\cdot\{2-g(x)\}dx$$

$$= 2\int_0^a f(x)\, dx - \int_0^a f(x)g(x)dx$$

$$\therefore 2\int_0^a f(x)f(x)\, dx = 2\int_0^a f(x)\, dx$$

$$\therefore \int_0^a f(x)\, g(x)\, dx = \int_0^a f(x)\, dx$$

11. Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ (1)

$$= \int_0^{\pi/2} f\left\{\sin 2\left(\frac{1}{2}\pi - x\right)\right\}\sin\left(\frac{1}{2}\pi - x\right)dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x\, dx \quad (2)$$

Then adding equations (1) and (2), we have

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

Now, from the result which we have to prove, it is clear that we have to substitute $\frac{\pi}{2} - 2\theta = 2x$.

$\therefore$ $dx = -d\theta$. Also, when $x = 0$, $\theta = \pi/4$, and when $x = \pi/2$, $\theta = -\pi/4$.

$$\therefore 2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\, d\theta$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\, d\theta$$

[as $g(\theta) = f(\cos 2\theta)\cos\theta$ is an even function]

$$\therefore I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx \quad (3)$$

Equations (1), (2), (3) give the required result.

12. $S = \cos x + \cos 3x + \ldots + \cos(2k-1)x$

$\therefore$ $(2\sin x)S = 2\sin x\cos x + 2\sin x\cos 3x + \ldots$
$+ 2\sin x\cos(2k-1)x$
$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \ldots$
$+ (\sin 2kx - \sin(2k-2)x)$
$= \sin 2kx$

$$\therefore \quad 2S = \frac{\sin 2kx}{\sin x}$$

$\Rightarrow$ $\sin 2kx \cot x$
$= \frac{\sin 2kx}{\sin x}\cos x$
$= 2\cos x(\cos x + \cos 3x + \ldots + \cos(2k-1)x)$

$= 1 + \cos 2x + \cos 4x + \cos 2x + \cos 6x + \cos 4x + \ldots +$
$\cos 2kx + \cos (2k-2)x$

$\therefore \int_0^{\pi/2} \sin 2kx \cot x dx$

$$= \int_0^{\pi/2} dx + \int_0^{\pi/2} (2\cos 2x + 2\cos 4x) + \cdots + 2\cos(2k-2)x$$

$$+ \int_0^{\pi/2} \cos 2kx = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

(As other integrals than 1st one are zero)

13. We are given that f is a continuous function and

$\int_0^x f(t)\, dt \to \infty$ as $|x| \to \infty$

To show that every line $y = mx$ intersects the curve

$y^2 + \int_0^x f(t)\, dt = 2$

if possible, let $y = mx$ intersects the given curve. Then substituting $y = mx$ in the curves, we get

$$m^2x^2 + \int_0^x f(t)\, dt = 2 \qquad (1)$$

Consider $F(x) = m^2x^2 + \int_0^x f(t)\, dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also, $F(x) \to \infty$ as $|x| \to \infty$

But $F(0) = -2$

Thus, $F(0) = -$ve and $F(b) = +$ve where b is some value of x and $F(x)$ is continuous.

Therefore, $F(x) = 0$ for some value of $x \in (0, b)$ or equation (1) is solvable for x.

Hence, $y = mx$ intersects the given curves.

14. Let $I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx \qquad (1)$

Then

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)}{2(\pi - x) - \pi} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\sin 2x)\sin\left(-\frac{\pi}{2}\cos x\right)}{\pi - 2x}, \text{or}$$

$$\text{or } I = \int_0^{\pi} \frac{(x - \pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx \qquad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \frac{(2x - \pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$$

$$= \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

$$= \int_0^{\pi} 2\sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

$$\text{or } I = \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

Put $z = \frac{\pi}{2}\cos x$. Then $dz = -\frac{\pi}{2}\sin x\, dx$.

When $x = 0$, $z = \frac{\pi}{2}$, and when $x = \pi$, $z = -\frac{\pi}{2}$.

$$\therefore I = -\frac{2}{\pi}\int_{\pi/2}^{-\pi/2} \frac{2z}{\pi} \sin z\, dz$$

$$= \frac{8}{\pi^2}\int_0^{\pi/2} z \sin z\, dz = \frac{8}{\pi^2}[-z\cos z + \sin z]_0^{\pi/2} = \frac{8}{\pi^2}$$

15. Given $\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$

where $n \in N$ and $n \le 5$.

To find the value of n, let

$$I_n = \int_0^1 e^x (x-1)^n\, dx$$

$$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1} e^x dx$$

$$= -(-1)^n - \int_0^1 n(x-1)^{n-1} e^x dx$$

$$\text{or } I_n = (-1)^{n+1} - nI_{n-1}. \qquad (1)$$

Also, $I_1 = \int_0^1 e^x (x-1)\, dx$

$$= \left[e^x (x-1)\right]_0^1 - \int_0^1 e^x\, dx$$

$$= -(-1) - (e^x)_0^1$$

$$= 1 - (e - 1) = 2 - e$$

Using equation (1), we get

$I_2 = (-1)^3 - 2I_1 = -1 - 2(2 - e) = 2e - 5$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1 - 3(2e - 5)$

$= 16 - 6e$

$\therefore \quad n = 3$

16. $I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx$

$$= \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$$

$$= \int_2^3 \frac{2x^3 dx}{(x^2+1)^2} + \int_2^3 \frac{1}{x^2-1} dx$$

$$= \int_2^3 \frac{x^2\, 2x dx}{(x^2+1)^2} + \left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$$

$$= \int_5^{10} \frac{t-1}{t^2} dt + \left[\frac{1}{2}\left(\log\frac{2}{4} - \log\frac{1}{3}\right)\right]$$

Put $x^2 + 1 = t$ or $2xdx = dt$. When $x \to 2$, $t \to 5$, and when $x \to 3$, $t \to 10$.

$$\therefore I = \int_5^{10}\left(\frac{1}{t} - \frac{1}{t^2}\right)dt + \frac{1}{2}\log\frac{3}{2}$$

$$= \left(\log|t| + \frac{1}{t}\right)_5^{10} + \frac{1}{2}\log\frac{3}{2}$$

$$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} + \frac{1}{2}\log\frac{3}{2}$$

$= \log 2 + \left(-\frac{1}{10}\right) + \frac{1}{2}\log\frac{3}{2}$

$= \frac{1}{2}\log 6 - \frac{1}{10}$

17. Let $I = \int_0^{n\pi+v} |\sin x|\,dx$

$= \int_0^{v}|\sin x|\,dx + \int_v^{n\pi+v}|\sin x|\,dx$

$= \int_0^{v}\sin x\,dx + n\int_0^{\pi}|\sin x|\,dx$ [$\because$ $|\sin x|$ has period π]

$= (-\cos x)_0^v + n(-\cos x)_0^{\pi}$

$= 2n + 1 - \cos v =$ R.H.S.

18. $U_{n+2} - U_{n+1} = \int_0^{\pi}\frac{(1-\cos(n+2)x)-(1-\cos(n+1)x)}{1-\cos x}dx$

$= \int_0^{\pi}\frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}$

$= \int_0^{\pi}\frac{2\sin\left(n+\frac{3}{2}\right)x\sin\frac{x}{2}}{2\sin^2 x/2}dx$

or $U_{n+2} - U_{n+1} = \int_0^{\pi}\frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}dx$ (1)

or $U_{n+1} - U_n = \int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$ (2)

From equations (1) and (2), we get

$\therefore (U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$

$= \int\frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$

or $U_{n+2} + U_n - 2U_{n+1} = \int\frac{2\cos(n+1)x\sin x/2}{\sin x/2}dx$

$= 2\int_0^{\pi}\cos(n+1)x\,dx = 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$

or $U_{n+2} + U_n = 2U_{n+1}$

$\therefore U_n, U_{n+1}, U_{n+2}$ are in A.P.

$U_0 = \int_0^{\pi}\frac{1-1}{1-\cos x}dx = 0,\ U_1 = \int_0^{\pi}\frac{1-\cos x}{1-\cos x}dx = \pi$

$U_1 - U_0 = \pi$ (Common difference)

$\therefore U_n = U_0 + n\pi = n\pi$

$\therefore U_n = n\pi$

Now, $I_n = \int_0^{\pi/2}\frac{\sin^2 n\theta}{\sin^2\theta}d\theta = \int_0^{\pi/2}\frac{1-\cos 2n\theta}{1-\cos 2\theta}d\theta$

$= \frac{1}{2}\int_0^{\pi}\frac{1-\cos nx}{1-\cos x}dx = \frac{1}{2}n\pi$

19. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x = -y$, so that $dx = -dy$

and $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}\cos^{-1}\left(\frac{-2y}{1+y^2}\right)dy$

But $\cos^{-1}(-x) = \pi - \cos^{-1}x$ for $-1 \le x \le 1$,

$\therefore I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}\left[\pi - \cos^{-1}\left(\frac{2y}{1+y^2}\right)\right]dy$

$= \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$= \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx - I$

$\therefore I = \pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx$

$= \pi\int_0^{1/\sqrt{3}}\left[-1 + \frac{1}{1-x^4}\right]dx$

$= -\pi\int_0^{1/\sqrt{3}}dx + \pi\int_0^{1/\sqrt{3}}\frac{dx}{1-x^4}$

$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\int_0^{1/\sqrt{3}}\left[\frac{1}{1-x^2} + \frac{1}{1+x^2}\right]dx$

$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\left(-\frac{1}{2}\log_e\left|\frac{1-x}{1+x}\right| + \tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$

$= -\frac{\pi}{\sqrt{3}} - \frac{\pi}{4}\log_e\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| + \frac{\pi^2}{12}$

20. Let $I = \int_0^{\pi/4}\ln(1+\tan x)dx$ (1)

$= \int_0^{\pi/4}\ln(1+\tan(\pi/4 - x))dx$

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/4}\ln\left[1 + \frac{1-\tan x}{1+\tan x}\right]dx$

$= \int_0^{\pi/4}\ln\left(\frac{2}{1+\tan x}\right)dx$

$\therefore I = \int_0^{\pi/4}[\ln 2 - \ln(1+\tan x)]dx$ (2)

Adding equations (1) and (2), we get

$2I = \int_0^{\pi/4}\ln 2\,dx$

$= \ln 2\,[x]_0^{\pi/4} = \ln 2\left[\frac{\pi}{4}\right]$

$\therefore I = \frac{\pi}{8}\ln 2$

21. Given $a + b = 4$ and $\frac{dg(x)}{dx} > 0$, i.e., $g(x)$ is an increasing function.

We have to prove that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b-a)$ increases.

Or we have to prove that $f(a) = \int_0^a g(x)dx + \int_0^{4-a} g(x)dx$ increases as $(4-2a)$ increases.

i.e., $\frac{df(a)}{d(4-2a)} > 0$

or $\frac{df(a)}{-2da} > 0$

or $\frac{df(a)}{da} < 0$

Now, $\frac{df(a)}{da} = g(a) - g(4-a)$

Given that $a < 2$

$\therefore \quad 2a < 4$

$\therefore \quad a < 4 - a$

$\therefore g(a) < g(4-a)$ (as given that $g(x)$ is increasing function)

Thus, $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b-a)$ increases.

22. $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + 2\int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \quad (1)$

$= 0 + 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}$

$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$

$= 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$\therefore 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

or $I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

Put $\cos x = t$ so that $-\sin x\, dx = dt$.
When $x = 0$, $t = 1$; when $x = \pi$, $t = -1$.

$\therefore I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2}$

$= 4\pi\left[\tan^{-1} t\right]_0^1$

$= 4\pi \frac{\pi}{4} = \pi^2$

23. $\int_0^1 \tan^{-1}\frac{1}{1-x+x^2} dx = \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)} dx$

$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\, dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}[1-(1-x)]\, dx$

$= 2\int_0^1 \tan^{-1} x\, dx \quad (1)$

Now,

$I = \int_0^1 \tan^{-1}(1-x+x^2)\, dx$

$= \int_0^1 \cot^{-1}\left(\frac{1}{1-x+x^2}\right) dx$

$= \int_0^1 \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{1-x+x^2}\right)\right] dx$

$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\, dx$ [From equation (1)]

$= \frac{\pi}{2} - 2\left\{x\tan^{-1} x - \frac{1}{2}\log(1+x^2)\right\}_0^1$ (integrating by parts)

$= \log_e 2$

24. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$

$= \int_1^x \frac{\log t}{1+t} dt + \int_1^{1/x} \frac{\log t}{1+t} dt$

In second integral, let $t = 1/y \therefore dt = -\frac{1}{y^2} dy$

$\therefore F(x) = \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{-\log y}{1+\frac{1}{y}}\left(-\frac{dy}{y^2}\right)$

$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log y}{y(1+y)} dy$

$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log t}{t(1+t)} dt$

$= \int_1^x \frac{\log t}{t} dt = \frac{1}{2}(\log x)^2$

$\therefore F(e) = \frac{1}{2}$

25. We have $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$

$= \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$

$\therefore \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x \frac{d}{dx}\left[\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta\right]$

$= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x\left[\frac{\cos x}{1+\sin^2 x} 2x - 0\right]$

or $\left.\frac{dy}{dx}\right|_{x=\pi} = 0 + \frac{\cos^2\pi}{1+\sin^2\pi}\cdot 2\pi = 2\pi$

26. Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$$

The second integral becomes zero as integrand being an odd function of x.

$$\therefore I = 2\pi \int_0^{\pi/3} \frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

Let $x + \pi/3 = y \therefore dx = dy$.
Also, as $x \to 0, y \to \pi/3$, and as $x \to \pi/3, y \to 2\pi/3$.

$$\therefore I = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2-\cos y}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2-\frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3\tan^2 y/2+1} dy$$

$$= \frac{4\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\frac{1}{2}\sec^2 y/2}{\tan^2 y/2+\left(1/\sqrt{3}\right)^2} dy$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}\left(\sqrt{3}\tan y/2\right)\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{4\pi}{\sqrt{3}}[\tan^{-1} 3 - \tan^{-1} 1]$$

$$= \frac{4\pi}{\sqrt{3}}[\tan^{-1} 3 - \pi/4]$$

27. $I = \int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x dx$

$$= \int_0^{\pi} e^{|\cos x|}\, 2\sin\left(\frac{1}{2}\cos x\right)\sin x\, dx + \int_0^{\pi} e^{|\cos x|} 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

$$= I_1 + I_2$$

Now, using the property that

$$\int_0^{2a} f(x)dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$$

we get $I_1 = 0$ and

$$I_2 = 2\int_0^{\pi/2} e^{|\cos x|}\, 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

$$= 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

Put $\cos x = t \therefore -\sin x\, dx = dt$

$$\therefore I_2 = 6\int_0^1 e^t \cos\frac{t}{2} dt$$

$$= 6\left[\frac{e^t}{1+\frac{1}{4}}\left(\frac{1}{2}\sin\frac{t}{2}+\cos\frac{t}{2}\right)\right]_0^1$$

$$\left[\text{Using } \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx)\right]$$

$$= \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$$

28. $\dfrac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx} = 5050\dfrac{I_{100}}{I_{101}}$

$$I_{101} = \int_0^1 (1-x^{50})(1-x^{50})^{100}dx$$

$$= I_{100} - \int_0^1 x\cdot x^{49}(1-x^{50})^{100}dx$$

$$= I_{100} - \left[\frac{-x(1-x^{50})^{101}}{101}\right]_0^1 - \int_0^1 \frac{(1-x^{50})^{101}}{5050}$$

$$\therefore I_{101} = I_{100} - \frac{I_{101}}{5050} \therefore 5050\frac{I_{100}}{I_{101}} = 5051$$